Matematika 1

7. vaja

B. Jurčič Zlobec[1]

[1] Univerza v Ljubljani,
Fakulteta za Elektrotehniko
1000 Ljubljana, Tržaška 25, Slovenija

Matematika FE, Ljubljana, Slovenija 4. januar 2013

Pravila za integriranje

I Linearnost:
$$\int (a\,f(x) + b\,g(x))\; dx = a \int f(x)\,dx + b \int g(x)\,dx.$$

II Per partes: $\displaystyle\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx.$

III Substitucija: $\displaystyle\int f(g(x))g'(x)\,dx = F(g(x))$, kjer je
$\int f(x)\,dx = F(x)$.
$u = g(x) \rightarrow, du = g'(x)\,dx \rightarrow, \int f(u)\,du = F(u) = F(g(x))$.

IV Linearna substitucija: $\int f(ax+b)\,dx = \frac{1}{a}F(ax+b)$.

Integrali elementarnih funkcij

1. $\int x^n\,dx = \frac{x^{n+1}}{n+1}, \quad n \in \mathbb{Z} \setminus \{-1\}, \quad x \neq 0.$
2. $\int x^s\,dx = \frac{x^{s+1}}{s+1}, \quad s \in \mathbb{R} \setminus \{-1\}, \quad x > 0.$
3. $\int e^x\,dx = e^x.$
4. $\int \frac{dx}{x} = \ln|x|.$
5. $\int \sin x\,dx = -\cos x.$
6. $\int \cos x\,dx = \sin x.$
7. $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\frac{x}{a}, \quad a > 0\ .$
8. $\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a}, \quad a > 0.$
9. $\int \frac{dx}{\sqrt{x^2\pm a^2}} = \ln(x + \sqrt{x^2 \pm a^2}), \quad a > 0.$

Reševanje integralov s pomočjo nastavka

A Integral oblike $\int f(x)\,dx = \int \frac{P(x)dx}{(x-a)^n(x^2+px+q)^m}$, kjer je $m, n \in \mathbb{N}$, $p^2-4q<0$ in $P(x)$ polinom stopnje manj ali enako $2m+n-1$, rešimo s pomočjo nastavka:
$$\int f(x)\,dx = \frac{Q_{2m-2+n-2}(x)}{(x-a)^{n-1}(x^2+px+q)^{m-1}} +$$
$$+A\ln|x-a| + B\ln(x^2+px+q) + C\operatorname{arctg}\frac{2x+p}{\sqrt{4q-p^2}}.$$

B Integral oblike $\int f(x)\,dx = \int \frac{P_n(x)\,dx}{\sqrt{x^2+px+q}}$ rešimo s pomočjo nastavka:
$$\int f(x)\,dx = Q_{n-1}(x)\sqrt{x^2+px+q} + \lambda\int\frac{dx}{\sqrt{x^2+px+q}}.$$

Določi $\int f(x)\,dx$, če je

$$f(x) = 3 + \frac{1}{x} + \frac{1}{x^2}.$$

- Uporabimo (I) $\rightarrow$ (1) $\rightarrow$ (4).
- $\int \left(3 + \frac{1}{x} + \frac{1}{x^2}\right)\, dx = \!\!\rightarrow$
- $\int 3\,dx + \int \frac{dx}{x} + \int \frac{dx}{x^2}\,dx = \!\!\rightarrow$
- $\displaystyle\int \left(3 + \frac{1}{x} + \frac{1}{x^2}\right) dx = 3x + \ln|x| - \frac{1}{x} + C.$

Določi $\int f(x)\,dx$, če je

$$f(x) = \frac{3+2x^2}{1+x^2}.$$

- Uporabimo (I) $\to$ (1) $\to$ (7).
- $\int \frac{3+2x^2}{1+x^2}\,dx = \int \frac{1+2(1+x^2)}{1+x^2}\,dx \to$
- $\int 2\,dx + \int \frac{dx}{1+x^2} = \to$
- $$\int \frac{3+2x^2}{1+x^2} dx = 2x + \operatorname{arctg} x + C.$$

Določi $\int f(x)\,dx$, če je

$$f(x) = \frac{x^3}{1+x^2}.$$

- Preverimo stopnji števca in imenovalca.
- Delimo: $\frac{x^3}{1+x^2} = x - \frac{x}{1+x^2}$
- (I) $\to$ (1) $\to$ (III) $\to$ (4).
- $u = 1 + x^2, du = 2x\,dx \to$
- $\frac{1}{2}x^2 - \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}x^2 - \frac{1}{2}\ln|u| \to$
- $\frac{1}{2}x^2 - \frac{1}{2}\ln(1+x^2) \to$
- $$\int \frac{x^3}{1+x^2} = \frac{1}{2}x^2 - \ln\sqrt{1+x^2} + C$$

Določi $\int f(x)\,dx$, če je

$$f(x) = \frac{3}{2+2x+x^2} + \frac{x}{1+x^2}.$$

- Uporabimo (I) $\to$ (IV) $\to$ (III) $\to$ (7) $\to$ (4).
- $\int \left(\frac{3dx}{2+2x+x^2} + \frac{x}{1+x^2} \right) dx = 3 \int \frac{dx}{(x+1)^2+1} + \int \frac{x}{1+x^2}\, dx \to$
- $u = x+1, du = dx, \quad v = 1+x^2, dv = 2x\,dx \to$
- $3 \int \frac{du}{1+u^2} + \frac{1}{2} \int \frac{dv}{v} = \to$
- $$\int f(x)\,dx = 3\,\mathrm{arctg}(x+1) + \frac{1}{2} \ln(1+x^2) + C.$$

Določi $\int f(x)\,dx$, če je

$$f(x) = \frac{x-1}{x(x+1)}.$$

- Razcepimo na parcialne ulomke. $\frac{x-1}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} = -\frac{1}{x} + \frac{2}{x+1}$.
- $\int \frac{x-1}{x(x+1)}\,dx = -\int \frac{dx}{x} + \int \frac{2dx}{x+1}$.
- $\int \frac{x-1}{x(x+1)}\,dx = -\ln|x| + 2\ln|x+1| \rightarrow$
- $$\int \frac{x-1}{x(x+1)}\,dx = \ln\frac{(x+1)^2}{|x|} + C.$$

Določi $\int f(x)\,dx$, če je

$$f(x) = \frac{x^2+1}{x(1-x^2)}.$$

- Razcepimo na parcialne ulomke.
  $\frac{x^2+1}{x(1-x^2)} = \frac{A}{x} + \frac{B}{1-x} + \frac{C}{1+x} \to$
- $x^2 + 1 = A(1-x)(1+x) + Bx(1+x) + Cx(1-x) \to$
- $-A + B - C = 1$, $B + C = 0$ in $A = 1$.
- $\displaystyle\int \frac{x^2+1}{x(1-x^2)}\,dx = \ln|x| - \ln|1-x^2|.$

Določi $\int f(x)\,dx$, če je

$$f(x) = \frac{x-1}{x(x^2+1)}.$$

- Razcepimo na parcialne ulomke.
  $\frac{x-1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = -\frac{1}{x} + \frac{1+x}{x^2+1}$.
- $\int \frac{x-1}{(x^2+1)}\,dx = -\int \frac{dx}{x} + \int \frac{dx}{1+x^2} + \int \frac{x\,dx}{1+x^2}$.
- $\int \frac{x-1}{x(x+1)}\,dx = -\ln|x| + \operatorname{arctg} x + \frac{1}{2}\ln(x^2+1) \to$
- $\displaystyle\int \frac{x-1}{x(x+1)}\,dx = -\ln|x| + \operatorname{arctg} x + \ln\sqrt{1+x^2} + C.$

Določi $\int f(x)\,dx$, če je

$$f(x) = \frac{x-1}{x^2(x+1)}.$$

- Razcepimo na parcialne ulomke.
  $\frac{x-1}{x^2(x+1)} = \frac{A}{x^2} + \frac{B}{x} + \frac{C}{x+1} = -\frac{1}{x^2} + \frac{2}{x} - \frac{2}{x+1}.$
- $\int \frac{x-1}{x(x+1)}\,dx = -\int \frac{dx}{x^2} + \int \frac{2dx}{x} - \int \frac{2dx}{1+x}.$
- $\int \frac{x-1}{x(x+1)}\,dx = \frac{1}{x} + 2\ln|x| - 2\ln|x+1| \rightarrow$
- $$\int \frac{x-1}{x(x+1)}\,dx = \frac{1}{x} + \ln\frac{x^2}{(1+x)^2} + C.$$

Določi $\int f(x)\,dx$, če je

$$f(x) = \frac{x^2+1}{x^2(1+x)}.$$

- Zapišemo nastavek (A).
- $\int \frac{x^2+1}{x^2(1+x)}\,dx = \frac{A}{x} + B\ln|x| + C\ln|1+x| \rightarrow$
- $x^2+1 = -A(1+x) + Bx(1+x) + Cx^2 \rightarrow$
- $B + C = 1$, $A + B = 0$ in $A = -1$.
- $$\int \frac{x^2+1}{x^2(1+x)}\,dx = -\frac{1}{x} - \ln|x| + 2\ln|1+x|.$$

Določi $\int f(x)\,dx$, če je

$$f(x) = \frac{x^3 - 1}{x^2(1+x)}.$$

- Stopnji števca in imenovalca sta enaki, delimo.
  $\frac{x^3-1}{x^2(1+x)} = 1 - \frac{x^2+1}{x^2(x+1)}$.
- Zapišemo nastavek (A).
  $\int \frac{x^2+1}{x^2(1+x)}\,dx = \frac{A}{x} + B\ln|x| + C\ln|1+x| \to$
- $x^2 + 1 = -A(1+x) + Bx(1+x) + Cx^2 \to$
- $B + C = 1$, $A + B = 0$ in $A = -1$.
- $\int \left(1 - \frac{x^2+1}{x^2(1+x)}\right) dx = x + \frac{1}{x} + \ln|x| - \ln(1+x)^2 + C.$

Določi $\int f(x)\,dx$, če je

$$f(x) = \frac{x^2-1}{x(2+2x+x^2)}.$$

- Preverimo stopnji števca in imenovalca.
- Zapišemo nastavek (A).
  $\int \frac{x^2-1}{x(2+2x+x^2)}\,dx = A\ln|x| + B\ln|x^2+x+1| + C\,\mathrm{arctg}(1+x) \to$
- $x^2-1 = \frac{A}{x} + \frac{2Bx+B+C}{x^2+x+1} \to$
- $B+C=1$, $A+B=0$ in $A=-1$.
- $$\int \frac{x^2-1}{x(x^2+2x+2)}\,dx =$$
  $$-\frac{1}{2}\ln|x| - \frac{1}{2}\,\mathrm{arctg}(x+1) + \frac{3}{4}\ln(2+2x+x^2) + C.$$

Določi $\int f(x)\,dx$, če je

$$f(x) = \frac{x}{\sqrt{x+1}}.$$

- (III) $\to$ (1).
- $\displaystyle\int \frac{x}{\sqrt{x+1}}\,dx \to u = x+1, du = dx \to$
- $\int \frac{u-1}{\sqrt{u}}\,du = \int \left(u^{\frac{1}{2}} - u^{-\frac{1}{2}}\right) du.$
- $\int \frac{x}{\sqrt{x+1}}\,dx = \frac{2}{3}\sqrt{(x+1)^3} - 2\sqrt{x+1} \to$
- $\displaystyle\int \frac{x}{\sqrt{x+1}}\,dx = \frac{2}{3}(x-2)\sqrt{1+x} + C$

Določi $\int f(x)\,dx$, če je

$f(x) = x\sqrt{1+x^2}$.

- (III) $\to$ (2).
- $\displaystyle\int x\sqrt{1+x^2} \to u = 1+x^2, du = 2x\,dx \to$
- $\frac{1}{2}\int u^{\frac{1}{2}}\,du = \frac{1}{3}u^{\frac{3}{2}}$.
- $\displaystyle\int x\sqrt{1+x^2} = \frac{1}{3}\sqrt{(1+x^2)^3} + C$.

Določi $\int f(x)\,dx$, če je

$f(x) = \sqrt{1-x^2}$.

- (III) $\to$ (2).
- $\displaystyle\int \sqrt{1-x^2}\,dx \to x = \sin u, dx = \cos u\,du \to$
- $\int \sqrt{1-\sin^2 u}\cos u\,du = \int \cos^2 u\,du = \to$
- $u + \frac{1}{2}\sin(2u) = u + \sin u \cos u = u + \sin u\sqrt{1-\sin^2 u}$.
- $\displaystyle\int \sqrt{1-x^2}\,dx = \frac{1}{2}\left(\arcsin x + x\sqrt{1-x^2}\right) + C$.

Določi $\int f(x)\,dx$, če je

$f(x) = \sqrt{1-x^2}$.

- Rešujemo s pomočjo nastavka (B).
- $\displaystyle\int \sqrt{1-x^2}\,dx = \rightarrow$
- $\displaystyle\int \frac{1-x^2}{\sqrt{1-x^2}}dx = (Ax+B)\sqrt{1-x^2} + C\int \frac{1}{\sqrt{1-x^2}}dx \rightarrow$
- $\frac{1-x^2}{\sqrt{1-x^2}} = A\sqrt{1-x^2} - \frac{(Ax+B)x}{\sqrt{1-x^2}} + \frac{C}{\sqrt{1-x^2}} \rightarrow$
- $1-x^2 = -2Ax^2 - Bx + C + A \rightarrow,\ A = C = \frac{1}{2},\ B = 0.$
- $\displaystyle\int \sqrt{1-x^2}\,dx = \frac{1}{2}\left(\arcsin x + x\sqrt{1-x^2}\right) + C.$

Določi $\int f(x)\,dx$, če je

$f(x) = \sqrt{1+x^2}$.

- Rešujemo s pomočjo nastavka (B).
- $\displaystyle\int \frac{1+x^2}{\sqrt{1+x^2}}dx = (Ax+B)\sqrt{1+x^2} + C\int \frac{1}{\sqrt{1+x^2}}dx \rightarrow$
- $\frac{1+x^2}{\sqrt{1+x^2}} = A\sqrt{1+x^2} + \frac{(Ax+B)x}{\sqrt{1+x^2}} + \frac{C}{\sqrt{1+x^2}} \rightarrow$
- $1+x^2 = 2Ax^2 + Bx + C + A \rightarrow$, $A = C = \frac{1}{2}$, $B = 0$.
- $\displaystyle\int \sqrt{1+x^2}\,dx = \frac{1}{2}\left(\ln(x+\sqrt{1+x^2}) + x\sqrt{1+x^2}\right) + C.$

Določi $\int f(x)\,dx$, če je

$$f(x) = \frac{x+2}{\sqrt{2+2x+x^2}}.$$

- Rešujemo s pomočjo nastavka (B).
- $\displaystyle\int \frac{x+2}{\sqrt{2+2x+x^2}}dx =$
  $\displaystyle A\sqrt{2+2x+x^2} + B\int \frac{dx}{\sqrt{2+2x+x^2}} \rightarrow$
- $\frac{x+2}{\sqrt{1+x^2}} = +\frac{A(x+1)}{\sqrt{2+2x+x^2}} + \frac{B}{\sqrt{2+2x+x^2}} \rightarrow$
- $x+2 = A(x+1) + B \rightarrow, A = 1, B = 1.$
- $\displaystyle\int \sqrt{x+2}\,dx =$
  $\sqrt{2+2x+x^2} + \ln(x+1+\sqrt{x^2+2x+2}) + C.$

Določi $\int f(x)\,dx$, če je

$$f(x) = \frac{x+2}{\sqrt{1+2x-x^2}}.$$

- Rešujemo s pomočjo nastavka (B).
- $\displaystyle\int \frac{x+2}{\sqrt{1+2x-x^2}}dx =$
  $\displaystyle A\sqrt{1+2x-x^2} + B\int \frac{dx}{\sqrt{1+2x-x^2}} \to$
- $\frac{x+2}{\sqrt{1+2x-x^2}} = \frac{A(2-2x)}{2\sqrt{1+2x-x^2}} + \frac{B}{\sqrt{1+2x-x^2}} \to$
- $x+2 = A - Ax + B$, $A = -1$ in $B = 3$.
- $\int \frac{1}{\sqrt{1+2x-x^2}}dx = \int \frac{dx}{\sqrt{2-(x-1)^2}} = \to(\text{IV})\to(8) = \arcsin(x-1).$
- $\displaystyle\int \frac{x+2}{\sqrt{2x-x^2}}dx = -\sqrt{1+2x-x^2} - 3\arcsin\frac{x-1}{\sqrt{2}} + C.$

Določi $\int f(x)\,dx$, če je

$f(x) = \dfrac{1}{x \ln x}$.

- (III) $\rightarrow$ (4).
- $\displaystyle\int \frac{dx}{x \ln x} \rightarrow u = \ln x, du = \frac{dx}{x} \rightarrow$
- $\int \frac{du}{u} = \ln u \rightarrow$
- $\displaystyle\int \frac{dx}{x \ln x} = \ln |\ln x| + C.$

Določi $\int f(x)\,dx$, če je

$$f(x) = \frac{e^x}{1+e^{2x}}.$$

- (III) $\to$ (7).
- $\displaystyle\int \frac{e^x}{1+e^{2x}}\,dx \to$
- $u = e^x, du = e^x\,dx \to$
- $\int \frac{du}{1+u^2} = \operatorname{arctg} u \to$
- $\displaystyle\int \frac{e^x}{1+e^{2x}}\,dx = \operatorname{arctg} e^x + C.$

Določi $\int f(x)\,dx$, če je

$f(x) = \operatorname{tg} x$.

- Upoštevamo, da je $\operatorname{tg} x = \dfrac{\sin x}{\cos x} \to$
- (III) $\to$ (4).
- $u = \cos x, dx = -\sin x\,dx \to$
- $\displaystyle\int \frac{\sin x}{\cos x} = \int \frac{du}{u} = -\ln|u| \to$
- $\displaystyle\int \operatorname{tg} x\,dx = -\ln|cosx| + C$

Določi $\int f(x)\,dx$, če je

$f(x) = \cos^2 x$.

- Upoštevamo, da je $\cos^2 x = \frac{1}{2}(1 + \cos(2x))$
- (I) $\to$ (1) $\to$ (IV) $\to$ (6).
- $\int \cos^2 x\,dx = \int dx + \int \cos(2x)\,dx = \to$
- $$\int \cos^2 x\,dx = \frac{x}{2} + \frac{1}{4}\sin(2x) + C.$$

Določi $\int f(x)\,dx$. če je

$f(x) = xe^{-x}$

- (II) $\rightarrow$ (1) $\rightarrow$ (IV) $\rightarrow$ (3).
- $\displaystyle\int xe^{-x}\,dx \rightarrow u = x, dv = e^{-x}\,dx, \quad du = dx, v = -e^{-x} \rightarrow$
- $-xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x}$
- $\displaystyle\int xe^{-x}\,dx = -e^{-x}(x+1) + C$

Določi $\int f(x)\,dx$, če je

$$f(x) = \frac{\sin x}{1 + \cos x}.$$

- $\displaystyle\int \frac{\sin x}{1 + \cos x}\,dx \to$
- $u = \cos x,\, du = -\sin x\,dx \to$
- $-\int \frac{du}{1+u} = -\ln|1 + u| = -\ln|1 + \cos x| \to$
- $\displaystyle\int \frac{\sin x}{1 + \cos x}\,dx = -\ln|1 + \cos x| + C$

Določi $\int f(x)\,dx$, če je

$f(x) = e^{-x}\cos(2x)$.

- Dvakrat uporabimo (II).
- $\int e^{-x}\cos(2x)\,dx \rightarrow$
- $u = e^{-x}, dv = \cos(2x)\,dx, \quad du = -e^{-x}\,dx, v = \frac{1}{2}\sin(2x)$
- $\int e^{-x}\cos(2x)\,dx = \frac{1}{2}e^{-x}\sin(2x) + \frac{1}{2}\int e^{-x}\sin(2x)\,dx$
- $\frac{1}{2}\int e^{-x}\sin(2x)\,dx \rightarrow$
- $u = e^{-x}, dv = \sin(2x)\,dx, \quad du = -e^{-x}, v = -\frac{1}{2}\cos(2x)$
- $\frac{1}{2}\int e^{-x}\sin(2x)\,dx = -\frac{1}{4}\left(e^{-x}\cos(2x) + \int e^{-x}\cos(2x)\right)$
- $\displaystyle\int e^{-x}\cos(2x)\,dx = \frac{1}{5}e^{-x}\left(2\sin(2x) - \cos(2x)\right) + C$

Določi $\int f(x)\,dx$, če je

$f(x) = x^2 \ln x$.

- Uporabimo (II).
- $$\int x^2 \ln x \, dx \to$$
- $u = \ln x, dv = x^2\,dx, \quad du = \frac{dx}{x}, v = \frac{x^3}{3} \to$
- $\int x^2 \ln x\,dx = \frac{1}{3}\left(x^3 \ln x - \int x^2\,dx\right)$.
- $$\int x^2 \ln x \, dx = \frac{x^3}{3} \ln x - \frac{x^3}{9} + C$$

Izračunaj določeni integral.

$$S = \int_0^4 e^{\sqrt{x}}\, dx.$$

- Uvedemo $t = \sqrt{x} \to t^2 = x \to 2t\,dt = dx$.
- Ker je $x = 0 \to t = 0$ in $x = 4 \to t = 2$, $S = 2\int_0^2 te^t\,dt$.
- Integriramo *per partes*:
  $u = t, \quad dv = e^t dt \to du = dt, \quad v = e^t$.
- $S = 2te^t\big|_0^2 - 2\int_0^2 e^t\,dt = 4e^2 - 2e^2 - 2 = 2(e^2 - 1)$.
- $\displaystyle\int_0^4 e^{\sqrt{x}} dx = 2(e^2 - 1)$.

Izračunaj določeni integral.

$$S = \int_0^2 xe^{x^2}\, dx.$$

- Uvedemo $t = x^2 \rightarrow dt = 2xdx \rightarrow dx = \frac{dt}{2}$.
- Ker je $x = 0 \rightarrow t = 0$ in $x = 2 \rightarrow t = 4$, $S = \frac{1}{2}\int_0^4 e^t dt$.
- $S = \dfrac{1}{2}\displaystyle\int_0^4 e^t dt = \dfrac{1}{2}(e^4 - 1)$.

Izračunaj določeni integral.

$$S = \int_0^{\pi^2} \sin\sqrt{x}\,dx.$$

- Uvedemo $t = \sqrt{x}, \to t^2 = x, \to 2tdt = dx$.
- Ker je $x = 0 \to t = 0$ in $x = \pi^2 \to t = \pi$, velja
  $S = 2\int_0^{\pi} t \sin t\,dt$.
- Integriramo *per partes*:
  $u = t, \quad dv = \sin t\,dt, \to du = dt, \quad v = -\cos t$.
- $S = -2t\cos t|_0^{\pi} + 2\int_0^{\pi} \cos t\,dt = \pi$

  $$S = 2\int_0^{\pi} t\sin t\,dt = 2\pi.$$

Izračunaj določeni integral.

$$S = \int_0^{\frac{\pi}{4}} \frac{dx}{\cos^2 x}.$$

- Uvedemo $t = \operatorname{tg} x$.
- $dt = \frac{dx}{\cos^2 x}$. Nove meje $x = 0 \to t = 0$, $x = \frac{\pi}{4} \to t = 1$.
- $\int_0^1 dt = t|_0^1 = 1$.

Izračunaj določeni integral.

$$\int_4^9 \frac{dx}{\sqrt{x}-1}.$$

- Uvedemo
  $x = t^2, \to dx = 2t\,dt, \quad x = 4 \to t = 2, \quad x = 9 \to t = 3.$
- $2\int_2^3 \frac{t\,dt}{t-1} = \; t + \ln|t-1||_2^3 = 2 + \ln 4.$
- $\displaystyle\int_4^9 \frac{dx}{\sqrt{x}-1} = 2 + \ln 4.$

Izračunaj določeni integral.

$$\int_{-1}^{2} \frac{dx}{9-x^2}.$$

- Razcepimo na parcialne ulomke: $\frac{1}{9-x^2} = \frac{1}{6}\left(\frac{1}{x-3} + \frac{1}{x+3}\right)$
- $\int_{-1}^{2} \frac{dx}{9-x^2} = \int_{-1}^{2} \frac{1}{6}\left(\frac{1}{x-3} + \frac{1}{x+3}\right) dx = \frac{1}{6} \ln |x^2 - 3| \Big|_{-1}^{2}.$
- $\displaystyle\int_{-1}^{2} \frac{dx}{9-x^2} = \frac{\ln 10}{6}.$

Dolokaži, da je $\int_0^\infty x^n e^{-x} dx = n!$.

- Uporabimo (III)
  $\to u = x^n, dv = e^{-x}, \quad du = nx^{n-1}, v = -e^{-x}$
- Pišimo $\Pi(n) = \int_0^\infty x^n e^{-x} dx$.
- $\int_0^\infty x^n e^{-x} dx = - x^n e^{-x}|_0^\infty + n \int_0^\infty x^{n-1} e^{-x} dx$
- Uporabimo l'Hôspitalovo pravilo pri računanju limite
  $\lim_{x\to\infty} x^n e^{-x} = \lim_{x\to\infty} \frac{x^n}{e^x} = 0$,
- in dobimo zvezo $\Pi(n) = n\Pi(n-1)$.
- Ker je $\Pi(0) = \int_0^\infty e^{-x} = -e^{-x}|_0^\infty = 1$,
- lahko z matematično indukcijo pokažemo, da je $\Pi(n) = n!$

Izračunaj (posplošeni) določeni integral.

$$\int_1^{\infty} \frac{dx}{x^2}.$$

- $\int_1^{\infty} \frac{dx}{x} = -\frac{1}{x}\Big|_1^{\infty} = 1.$
- $\int_0^{\infty} \frac{dx}{x^2} = 1.$

Izračunaj (posplošeni) določeni integral.

$$\int_0^4 \frac{dx}{\sqrt{4-x}}.$$

- Uvedemo $t^2 = 4 - x \to 2t\,dt = -dx$.
- Meje $x = 0 \to t = 2, \quad x = 4 \to t = 0$.
- $-2\int_2^0 \frac{t\,dt}{t} = -2t|_2^0 = 4$
- $\int_0^4 \frac{dx}{\sqrt{4-x}} = 4$

Izračunaj (posplošeni) določeni integral.

$$\int_{-2}^{1} \frac{|x|}{x} dx.$$

- $\int_{-2}^{1} \frac{|x|}{x}\, dx = \int_{-2}^{0} (-1) dx + \int_{0}^{1} dx = -x|_{-2}^{0} + x|_{0}^{1} = -1$
- $\int_{-2}^{1} \frac{|x|}{x} dx = -1$

Izračunaj (posplošeni) določeni integral.

$$\int_0^\infty e^{-\sqrt{x}} dx.$$

- Uvedemo $t^2 = x \to 2t\, dt = dx$.
- Meje $x = 0 \to t = 0, \quad x \to \infty \to t \to \infty$.
- $\int_0^\infty e^{-\sqrt{x}} dx = 2 \int_0^\infty te^{-t}\, dt$
- Integriramo *per partes*:
  $du = e^{-t} dx, v = t \to u = -e^{-t}, dv = dt$.
- $2 \int_0^\infty te^{-t}\, dt = -2te^{-t}\Big|_0^\infty + 2 \int_0^\infty e^{-t} dt$
- Ker je $\lim_{t\to\infty} te^{-t} = 0$, je $2 \int_0^\infty te^{-t} dt = 2$.
- $$\int_0^\infty e^{-\sqrt{x}} dx = 2.$$

Izračunaj (posplošeni) določeni integral.

$$\int_0^\infty \frac{e^{-\sqrt{x}}dx}{\sqrt{x}}.$$

- Uvedemo $t^2 = x \rightarrow 2t\,dt = dx$.
- Meje $x = 0 \rightarrow t = 0, \quad x \rightarrow \infty \rightarrow t \rightarrow \infty$.
- $\int_0^\infty \frac{e^{-\sqrt{x}}dx}{\sqrt{x}} = 2\int_0^\infty e^{-t}\,dt = -2e^{-t}\Big|_0^\infty = 2.$
- $\int_0^\infty \frac{e^{-\sqrt{x}}dx}{\sqrt{x}} = 2.$

Izračunaj (posplošeni) določeni integral.

$$\int_0^1 \sqrt{x} \ln \frac{1}{x}\, dx.$$

- $\int_0^1 \sqrt{x} \ln \frac{1}{x}\, dx = -\int_0^1 \sqrt{x} \ln x \, dx$
- Integriramo *per partes*.
  $u = \ln x, dv = \sqrt{x}\, dx \rightarrow du = \frac{dx}{x}, v = \frac{2}{3}\sqrt{x^3}$
- $$\int_0^1 \sqrt{x} \ln x \, dx = \frac{2}{3}\sqrt{x^3} \ln x \Big|_0^1 - \frac{2}{3} \int_0^1 \sqrt{x}\, dx$$
- $$\int_0^1 \sqrt{x} \ln \frac{1}{x}\, dx = \frac{4}{9}$$