Matematika 1

8. vaja

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Parametrična oblika

$$x = x(t), y = y(t), \quad t \in [t_1, t_2] \subset \mathbb{R}.$$

1. Funkciji $x = x(t)$ in $y = y(t)$ sta zvezni in odvedljivi definirani na $[t_1, t_2]$.
2. Smerni koeficient tangente $\frac{dy}{dx} = \frac{\dot{y}(t)}{\dot{x}(t)}$.
3. Ploščina zanke $S = \frac{1}{2}\int_{t_1}^{t_2} (x(t)\dot{y}(t) - y(t)\dot{x}(t))\,dt$.
4. Dolžina loka $s = \int_{t_1}^{t_2} \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}dt$.

Eksplicitna oblika $y = f(x), \quad x \in [x_1, x_2] \subset \mathcal{D}_f \subset \mathbb{R}$.

- Kot poseben primer parametrične oblike $x = x$, $y = f(x)$.
- Smerni koeficient tangente $\dfrac{dy}{dx} = f'(x)$.
- Ploščina zanke, ki jo določa krivulja in os x
  $S = \displaystyle\int_{x_1}^{x_2} f(x)\, dx$.
- Dolžina loka $s = \displaystyle\int_{x_1}^{x_2} \sqrt{1 + f'(x)^2} dx$.

Polarna oblika $r = r(\varphi), \quad \varphi \in [\varphi_1, \varphi_2] \subset \mathbb{R}$

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- Kot poseben primer parametrične oblike $x = r(\varphi)\cos\varphi$, $y = r(\varphi)\sin\varphi$.
- Smerni koeficient tangente $$\frac{dy}{dx} = \frac{r(\varphi)\cos\varphi + r'(\varphi)\sin\varphi}{-r(\varphi)\sin\varphi + r'(\varphi)\cos\varphi}.$$
- Ploščina območja, ki jo določa krivulja in kota φ_1, φ_2 je $$S = \frac{1}{2}\int_{\varphi_1}^{\varphi_2} r(\varphi)^2\, d\varphi.$$
- Dolžina loka med kotoma φ_1 in φ_2 je $$s = \int_{\varphi_1}^{\varphi_2} \sqrt{r(\varphi)^2 + r'(\varphi)^2} d\varphi.$$

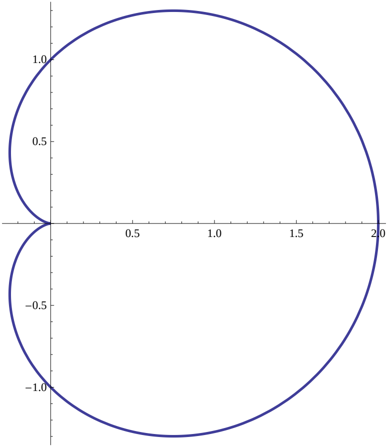
Implicitna oblika $f(x, y) = 0$.

- Smerni koeficient tangente. Odvajamo implicitno. $\frac{d}{dx}f(x, y(x)) = 0$ in izrazimo $y'(x)$.
  - Primer $x^2 + xy + \frac{y^2}{x} = 0, \rightarrow$
  - $2x + y + xy' + 2yy'x + y^2 = 0, \rightarrow$
  - $y' = -\dfrac{2x + y + y^2}{x + 2xy}$
- Večinoma bomo imeli opravka s primeri, ko z uvedbo polarnih koordinat, implicitno obliko prevedemo na polarno.

Graf srčnice $(x^2+y^2-x)^2=(x^2+y^2)$

Vpeljemo polarne koordinate: $r^2=x^2+y^2$, $x=r\cos\varphi$ in dobimo $r=1+\cos\varphi$.

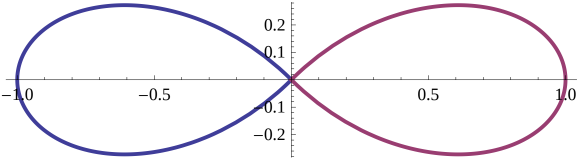
```
PolarPlot[1+Cos[t],{t,0,2Pi},PlotStyle->Thick]
```



Graf lemniskate $(x^2 + y^2)^2 = (x^2 - y^2)$

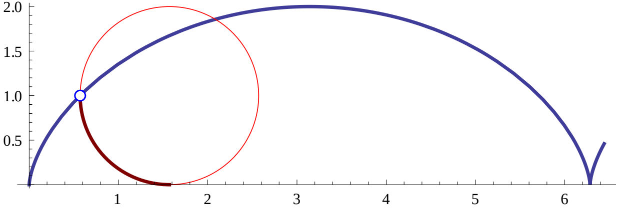
Vpeljemo polarne koordinate in dobimo
$r = \sqrt{\cos(2\varphi)}, \quad -\frac{\pi}{4} \le \varphi \le \frac{\pi}{4},$

```
PolarPlot[Sqrt[Cos[2t]],{t,-Pi/4,Pi/4},
PlotStyle->Thick]
```



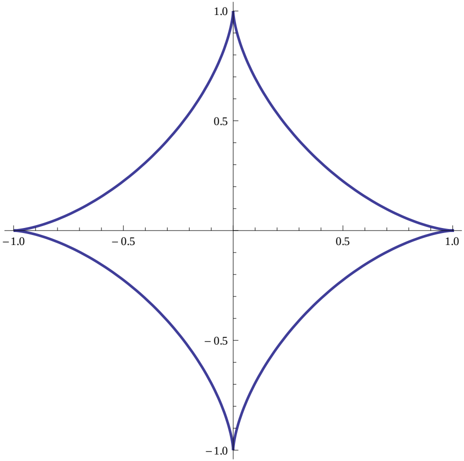
Graf cikloide $x(t) = t - \sin t, \quad y(t) = 1 - \cos t.$

```
ParametricPlot[t-Sin[t],1-Cos[t],{t,0,2Pi},
PlotStyle->Thick]
```



Graf asteroide $x(t) = \cos^t, \quad y(t) = \sin^3 t.$

```
ParametricPlot[Cos[t]^3,Sin[t]^3,{t,0,2Pi},
PlotStyle->Thick]
```

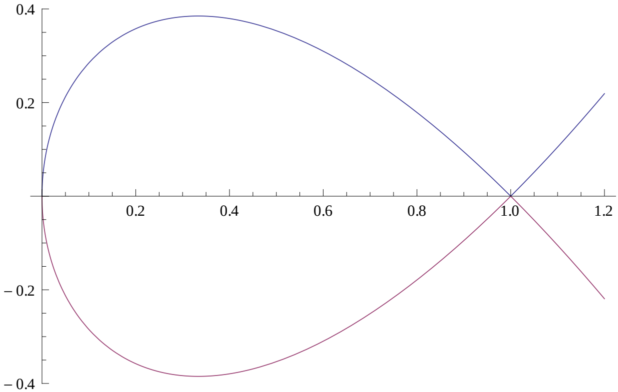


Nariši graf krivulje $y^2 = x(1-x)^2$.

- Krivulja je sestavljena iz dveh delov.
- $y = \pm\sqrt{x}|1-x|$, $x > 0$.

```
Plot[Sqrt[x]*Abs[1-x],-Sqrt[x]*Abs[1-x],x,0,1.2];
```

Graf

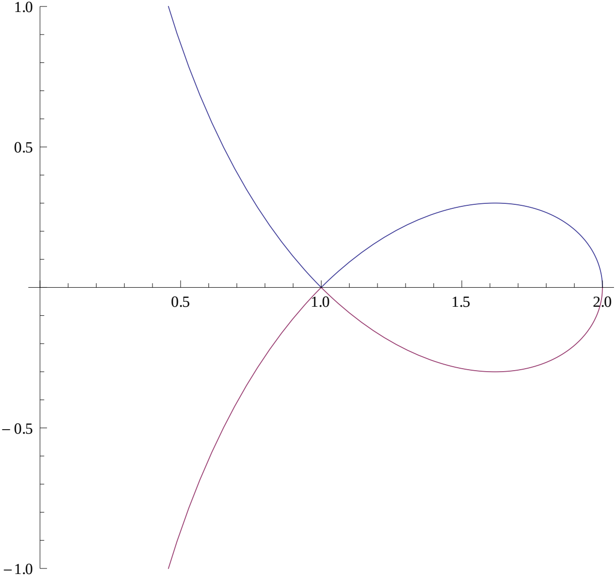


Nariši graf krivulje $y^2 = \dfrac{x}{2-x}(1-x)^2$.

- Krivulja je sestavljena iz dveh delov.
- $y = \pm\sqrt{\frac{x}{2-x}}|1-x|, x > 0.$

```
Plot[Sqrt[x/(2-x)]*Abs[1-x],-Sqrt[x/(2-x)]*Abs[1-x]
```

Graf



Izračunaj ploščino zanke $y^2 = x(1-x)^2$.

- $S = 2\int_0^1 \sqrt{x}(1-x)\,dx \rightarrow$.
- $2\,\frac{2}{15}x^{3/2}(-5+3x)\Big|_0^1$.
- $S = \frac{4}{15}$.

Izračunaj ploščino pod lokom cikloide $x = t - \sin t, y = 1 - \cos t$.

- $S = \int_0^{2\pi} y(t)\dot{x}(t)dt, \rightarrow$
- $\int_0^{2\pi} (1 - \cos t)^2 dt, \rightarrow$
- $\frac{3t}{2} - 2\sin t + \frac{1}{4}\sin(2t)\Big|_0^{2\pi}$.
- $S = 3\pi$.

Izračunaj ploščino srčnice $r(\varphi) = 1 + \cos\varphi$.

- $S = \frac{1}{2}\int_0^{2\pi} (1 + \cos\varphi)^2 d\varphi, \rightarrow$
- $\int_0^{2\pi} (1 + 2\cos\varphi + \cos^2\varphi) d\varphi, \rightarrow$
- $\frac{3t}{2} + 2\sin t + \frac{1}{4}\sin(2t)\Big|_0^{2\pi}.$
- $S = 3\pi.$

Izračunaj ploščino lemniskate $(x^2 + y^2)^2 = x^2 - y^2$.

- Polarna oblika $r(\varphi) = \sqrt{\cos(2\varphi)}$.
- $2\int_{-\pi/4}^{\pi/4} \cos(2\varphi) d\varphi$.
- $S = 2$.

Izračunaj ploščino asteroide $x^{2/3} - y^{2/3} = 1$.

- Parametrična oblika $x(t) = \cos^3 t, y(t) = \sin^3 t$.
- $S = \frac{1}{2}\int_0^{2\pi} (x(t)\dot{y}(t) - y(t)\dot{x}(t))\, dt \rightarrow$
- $3\int_0^{2\pi} \sin^2 t \cos^2 t\, dt \rightarrow \frac{3}{4}\int_0^{2\pi} \sin^2(2t)\, dt \rightarrow$
  $\frac{3}{8}\int_0^{2\pi} (1 - \cos(4x)\, dx$
- $S = 3\,\frac{t}{8} - \frac{1}{32}\sin(4t)\Big|_0^{2\pi}$, $S = \frac{3\pi}{4}$.

Izračunaj dolžino enega loka cikloide $x(t) = t - \sin t, y(t) = 1 - \cos t, t \in [0, 2\pi]$.

- $s = \int_0^{2\pi} \sqrt{\dot{x}^2(t) + \dot{y}^2(t)} dt, \rightarrow$
- $\int_0^{2\pi} \sqrt{2 - 2\cos t} dt, \rightarrow$
- $\int_0^{2\pi} 2|\sin(t/2)| dt = 4\cos(t/2) \Big|_0^{2\pi}.$
- $s = 8.$

Izračunaj dolžino krivulje $r(\varphi) = \frac{1}{\cos\varphi}, \varphi \in [0, \frac{\pi}{4}]$.

- $x(\varphi) = r(\varphi)\cos\varphi = 1,$
- $y(\varphi) = r(\varphi)\sin\varphi = \tan\varphi.$
- $S = \int_0^{\pi/4} (1 + \tan^2\varphi) d\varphi = \tan\varphi \Big|_0^{\pi/4} \to S = 1.$

Prostornina in površina vrtenine, parametrična oblika.

Vrtenina je dobljena z vrtenjem krivulje $y = f(x)$ okoli osi x za $x_1 \leq x \leq x_2$.

- Prostornina
$$V = \pi \int_{x_1}^{x_2} f(x)^2 dx$$
- Površina
$$P = 2\pi \int_{x_1}^{x_2} f(x)\sqrt{1 + f'(x)^2}\, dx$$

Prostornina in površina vrtenine, parametrična oblika.

Vrtenina je dobljena z vrtenjem krivulje $x = x(t)$, $y = y(t)$ okoli osi x za $t_1 \leq t \leq t_2$.

- Prostornina
$$V = \pi \int_{t_1}^{t_2} y(t)^2 \dot{x}(t)\, dt$$
- Površina
$$P = 2\pi \int_{t_1}^{t_2} y(t) \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2}\, dt$$