Univerza *v Ljubljani*
Fakulteta *za elektrotehniko*
Fakulteta *za računalništvo in informatiko*

ZBIRKA NALOG IZ MATEMATIKE III

Založba FE in FRI

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UNIVERZA V LJUBLJANI
FAKULTETA ZA ELEKTROTEHNIKO

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VSEBINA

DIFERENCIALNA GEOMETRIJA

Parametrična enačba krivulje	$\vec{r} = (x(t)\,,\, y(t)\,,\, z(t))$
smer tangente	$\vec{r}\,' = (x'(t)\,,\, y'(t)\,,\, z'(t))$
dolžina loka	$s = \int\limits_{t_0}^{t} \sqrt{x'^2 + y'^2 + z'^2}\, dt$
Parametrična enačba ploskve	$\vec{r} = (x(u,v)\,,\, y(u,v)\,,\, z(u,v))$
smer normale	$\vec{\nu} = \vec{r}_u \times \vec{r}_v$
Eksplicitna enačba ploskve	$z = f(x,y)$
smer normale	$\vec{\nu} = (\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1)$
Implicitna enačba ploskve	$F(x,y,z) = 0$
smer normale	$\vec{\nu} = (\frac{\partial F}{\partial x}\,,\, \frac{\partial F}{\partial y}\,,\, \frac{\partial F}{\partial z})$

1. Zapiši enačbo krivulje v parametrični obliki:

   a) daljica od točke $A(1,5,3)$ do točke $B(0,2,-1)$

   b) parabola $y = x^2$ v višini $z = 1$

   c) elipsa $x^2 + y^2 = 4$, $z = x + y$

   d) presečišče sfere $x^2+y^2+z^2 = 1$ z ravnino $x = y$ v prvem oktantu

   e) presečišče sfere $x^2 + y^2 + z^2 = R^2$ z valjem $x^2 + y^2 = \frac{R^2}{4}$

   f) presečišče sfere $x^2 + y^2 + z^2 = R^2$ z valjem $x^2 + y^2 = Rx$ v zgornjem polprostoru $z \geq 0$

2. Izračunaj dolžino loka prostorskih krivulj:

   a) $\vec{r} = (2t\,,\, \ln t\,,\, t^2)$, $1 \leq t \leq 10$

   b) $\vec{r} = (3t\,,\, 3t^2\,,\, 2t^3)$ od točke $A(0,0,0)$ do točke $B(3,3,2)$

   c) $\vec{r} = (t \sin t + \cos t\,,\, -t \cos t + \sin t\,,\, \frac{\sqrt{3}}{2} t^2)$, $0 \leq t \leq a$

d) $\vec{r} = (2\cos t\,,\, 2\sin t\,,\, \frac{3t}{\pi})$, $0 \le t \le \pi$

e) $\vec{r} = (\sin^2 t\,,\, \sin t \cos t\,,\, \ln\cos t)$, $0 \le t \le \frac{\pi}{3}$

f) $\vec{r} = (t + \frac{1}{t}\,,\, t - \frac{1}{t}\,,\, 2\ln t)$, $1 \le t \le 2$

g) $\vec{r} = (a\cos t\,,\, a\sin t\,,\, a\ln\cos t)$

od točke $A(a,0,0)$ do točke $B(\frac{a}{\sqrt{2}}, \frac{a}{\sqrt{2}}, -\frac{a\ln 2}{2})$

h) $\vec{r} = (e^t\cos t\,,\, e^t\sin t\,,\, e^t)$ od $A(1,0,1)$ do splošne točke

i) $x^2 = 3y$, $2xy = 9z$ od $A(0,0,0)$ do $B(3,3,2)$

j) $z^2 = 6x$, $9y^2 = 16xz$ od $A(0,0,0)$ do $B(6,8,6)$

k) $4ax = (y+z)^2$, $4x^2 + 3y^2 = 3z^2$ od izhodišča do $T(x,y,z)$

l) $y = a\arcsin\frac{x}{a}$, $z = \frac{a}{4}\ln\frac{a+x}{a-x}$

od izhodišča do točke $(\frac{a}{2}, \frac{\pi a}{6}, \frac{a}{4}\ln 3)$

m) $y = \frac{x^2}{2}$, $z = \frac{x^3}{6}$ od $x = 0$ do $x = 6$

3. Izrazi krivuljo z naravnim parametrom:

a) $\vec{r} = (a\cos t\,,\, a\sin t\,,\, b\,t)$

b) $\vec{r} = (e^t\cos t\,,\, e^t\sin t\,,\, e^t)$

c) $\vec{r} = (3t\,,\, 3\sqrt{1-t^2}\,,\, 4\arccos t)$

4. Prepričaj se, da je krivulja

$$\vec{r} = \Big(\frac{s+\sqrt{s^2+1}}{2}\,,\ \frac{1}{2(s+\sqrt{s^2+1})}\,,\ \frac{1}{\sqrt{2}}\ln(s+\sqrt{s^2+1})\Big)$$

izražena z naravnim parametrom, tj. $|\frac{d\vec{r}}{ds}| = 1$!

5. Prepričaj se, da je krivulja $\vec{r} = (1+\cos t\,,\, \sin t\,,\, 2\sin\frac{t}{2})$ presečišče sfere $x^2+y^2+z^2 = 4$ in valja $(x-1)^2+y^2 = 1$!

6. Presečišče valja $x^2+y^2 = 1$ in ravnine $x+y+z = 1$ je elipsa. Poišči njeno parametrično enačbo!

7. Krivulja $\vec{r} = (-2+\sin t\,,\, t^2+2\,,\, t^2-1+2\sin t)$ leži v neki ravnini. Poišči to ravnino!

8. $\vec{r} = (\,a\cos^2 t\,,\ a\sqrt{2}\sin t\cos t\,,\ a\sin^2 t\,)\ ,\ \ 0 \le t \le \pi$ je parametrična enačba krožnice v prostoru. Poišči njeno lego!

9. Poišči enačbo tangentne premice in normalne ravnine na krivuljo v dani točki:

a)	$\vec{r} = (a\sin^2 t\,,\ b\sin t\cos t\,,\ c\cos^2 t)$	v točki $t = \frac{\pi}{4}$
b)	$\vec{r} = (t - \sin t\,,\ 1 - \cos t\,,\ 4\sin\frac{t}{2})$	v točki $(\frac{\pi}{2} - 1\,,\ 1\,,\ 2\sqrt{2})$
c)	$\vec{r} = (t^3 - t^2 - 5\,,\ 3t^2 + 1\,,\ 2t^3 - 16)$	v točki $t = 2$
d)	$\vec{r} = (t\,,\ t^2\,,\ t^3)$	v točki $(2, 4, 8)$
e)	$\vec{r} = (\frac{t^4}{4}\,,\ \frac{t^3}{3}\,,\ \frac{t^2}{2})$	v poljubni točki
f)	$y = x\ ,\ z = x^2$	v točki $(1, 1, 1)$
g)	$x^2 + z^2 = 10\ ,\ y^2 + z^2 = 10$	v točki $(1, 1, 3)$
h)	$x^2 + y^2 + z^2 = 6\ ,\ x + y + z = 0$	v točki $(1, -2, 1)$
i)	$y^2 + z^2 = 25\ ,\ x^2 + y^2 = 10$	v točki $(1, 3, 4)$
j)	$x^2 + y^2 = z^2\ ,\ x = y$	v točki $(1, 1, -\sqrt{2})$
k)	$x^2 + y^2 + z^2 = 3\ ,\ x^2 + y^2 = 2$	v točki $(1, 1, 1)$

10. Poišči točko na krivulji, v kateri je tangenta vzporedna dani ravnini:

a)	$\vec{r} = (\frac{t^4}{4}\,,\ \frac{t^3}{3}\,,\ \frac{t^2}{2})$	$x + 3y + 2z - 10 = 0$
b)	$\vec{r} = (t\,,\ t^2\,,\ t^3)$	$x + 2y + z = 4$
c)	$\vec{r} = (\ln t\,,\ 2t\,,\ t^2)$	$8x - 4y + z = 0$

11. Pokaži, da tangenta na krivuljo $\vec{r} = (t\,,\ \frac{1}{3}t^2\,,\ \frac{2}{27}t^3)$ oklepa konstanten kot z nekim vektorjem! Poišči ta vektor!

12. Poišči točko na krivulji $\vec{r} = (t^2 + 1, t, t^2 + 2)$. ki je najbližja koordinatnemu izhodišču!

13. Zapiši dano ploskev v parametrični obliki:

a) $x^2 + y^2 = R^2 \quad , \quad 0 \le z \le H$

b) $z = \sqrt{x^2 + y^2}$

c) paraboličen valj $y = x^2$

d) ravnina $z = x$

14. Zapiši enačbo dane ploskve v kartezičnih koordinatah:

a) $\vec{r} = (u+v\,,\ u-v\,,\ u^2+v^2)$

b) $\vec{r} = (v\cos u\,,\ v\sin u\,,\ \sin 2u)$

c) $\vec{r} = (u\cos v\,,\ u\sin v\,,\ \sqrt{a^2-u^2})$

15. Ugotovi obliko ploskve! Kaj so koordinatne krivulje $u =$ konst. in $v =$ konst. ?

a)	$\vec{r} = (u\cos v, u\sin v, \sqrt{a^2-u^2})$	$0 \le u \le a\,,\ 0 \le v \le 2\pi$
b)	$\vec{r} = (u\cos v\,,\ u\sin v\,,\ v)$	$0 \le u \le 1\,,\ -\infty < v < \infty$
c)	$\vec{r} = (u\,,\ v\,,\ 0)$	$-\infty < u < \infty\,,\ -\infty < v < \infty$
d)	$\vec{r} = (u\cos v\,,\ u\sin v\,,\ 0)$	$0 \le u < \infty\,,\ 0 \le v \le 2\pi$
e)	$\vec{r} = (\cos u\,,\ \sin u\,,\ v)$	$0 \le u \le 2\pi\,,\ 0 \le v \le H$
f)	$\vec{r} = (\cos u\,,\ v\,,\ \sin u)$	$0 \le u \le \pi\,,\ 0 < v < \infty$

16. Poišči kot, pod katerim se sekata koordinatni krivulji $u = 1$ in $v = 2$ na ploskvi $\vec{r} = (u+v, u-v, uv)$!

17. Izračunaj kot, pod katerim se sekajo koordinatne krivulje na ploskvi $\vec{r} = (u\sin v, u\cos v, v)$!

18. Poišči kot, ki ga oklepa dana krivulja na ploskvi s predpisano koordinatno krivuljo:

a) ploskev $\vec{r} = (u\cos v\,,\ u\sin v\,,\ u)$

krivulja $\vec{r} = (e^t\cos t\,,\ e^t\sin t\,,\ e^t)$

koordinatna krivulja $v = v_0$

b) ploskev $\vec{r} = (\cos u\cos v\,,\ \sin u\cos v\,,\ \sin v)$

krivulja $\vec{r} = (\cos^2 u\,,\ \sin u\cos u\,,\ \sin u)$

koordinatna krivulja $u = 0$

19. Zapiši prvo fundamentalno formo za naslednje ploskve:

a) $\vec{r} = (u + v\,,\ u - v\,,\ uv)$

b) $\vec{r} = (u \sin v\,,\ u \cos v\,,\ v)$

c) $\vec{r} = ((b - a \cos u) \cos v\,,\ (b - a \cos u) \sin v\,,\ a \sin u) \quad , \quad b > a > 0$

d) $\vec{r} = (u\,,\ v\,,\ u^2)$

20. $ds^2 = du^2 + u^2 dv^2$ je prva fundamentalna forma za ploskev $\vec{r}(u, v)$. Poišči kot med krivuljama $v = u$ in $v = 1$!

21. Določi kot med ploskvama $x^2 + y^2 - z^2 = 0$ in $xz + yz = 0$!

22. Poišči točke na ploskvi $\vec{r}(u, v) = (\sqrt{u} \cos v\,,\ \sqrt{u} \sin v\,,\ \frac{2}{u})$, ki so najbljižje izhodišču!

23. Zapiši enačbi tangentne ravnine in normalne premice na dano ploskev:

a) $z = y + \ln \frac{x}{z}$ v točki $T(1, 1, 1)$

b) $z = x^2 + y^2$ v točki $T(1, 2, 5)$

c) $z = \operatorname{arctg} \frac{y}{x}$ v točki $T(1, 1, \frac{\pi}{4})$

d) $x^2 + y^2 + z^2 = 169$ v točki $T(3, 4, 12)$

e) $2^{x/z} + 2^{y/z} = 8$ v točki $T(2, 2, 1)$

f) $x^3 + y^3 + z^3 + xyz - 6 = 0$ v točki $T(1, 2, -1)$

g) $(z^2 - x^2)xyz - y^5 = 5$ v točki $T(1, 1, 2)$

h) $\vec{r} = (u \cos v, u \sin v, \sqrt{3}v)$ v točki $T(\frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{\pi}{\sqrt{3}})$

i) $\vec{r} = (u^2 + v^2, u - v, 4uv)$ v točki $T(5, 3, -8)$

24. Na ploskvi $\vec{r} = (2 \cos u, v, 2 \sin u)$, $0 < u < \pi$ določi vse točke, v katerih je normalni vektor navpičen!

25. K elipsoidu $x^2 + 2y^2 + z^2 = 1$ poišči tangentno ravnino vzporedno ravnini $x - y + 2z = 0$!

26. Poišči tangentno ravnino na ploskev $z = xy$ pravokotno na premico $\frac{x+2}{2} = y + 2 = \frac{z-1}{-1}$!

27. Na ploskvi $x^2+2y^2+3z^2+2xy+2xz+4yz=8$ poišči točke, v katerih je tangentna ravnina vzporedna ravnini $x=0$!

28. Poišči enačbo tangentne ravnine na ploskev $\vec{r}=(uv, \frac{u}{v}, \frac{1}{u^2})$ v tisti točki na ploskvi, tako da bo tangentna ravnina vzporedna ravnini $x+y+z=0$!

29. Poišči točko T v prvem oktantu na elipsoidu $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, ki ima naslednjo lastnost: normalna premica na elipsoid v točki T tvori s koordinatnimi osmi enake kote!

30. Poišči tangentno ravnino na ploskev $x^2+2y^2+3z^2=21$ vzporedno ravnini $x+4y+6z=0$!

31. Pod kakšnim kotom se sekata valj $x^2+y^2=1$ in ploskev $z=xy$ v skupni točki $T(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{2})$?

32. Pod kakšnim kotom se sekata sfera $(x-R)^2+y^2+z^2=R^2$ in valj $x^2+y^2=R^2$ v točki $T(\frac{R}{2}, \frac{R\sqrt{3}}{2}, 0)$?

33. Tangentna ravnina na ploskev $xyz=a^3$ ($a>0$) v neki izbrani točki tvori tetraeder skupaj s koordinatnimi ravninami $x=0$, $y=0$ in $z=0$. Dokaži, da je volumen tega tetraedra neodvisen od izbrane točke na ploskvi!

34. Dokaži, da tangentna ravnina na ploskev $\sqrt{x}+\sqrt{y}+\sqrt{z}=\sqrt{a}$ $(a>0)$ v točki T odreže na koordinatnih oseh odseke, katerih vsota je neod–visna od točke T!

INTEGRALI S PARAMETROM

$$F(x) = \int_{u(x)}^{v(x)} f(x,y)\, dy$$

$$F'(x) = \int_{u(x)}^{v(x)} \frac{\partial f}{\partial x}\, dy \; + f(x, v(x))\, v'(x) - f(x, u(x))\, u'(x)$$

1. Poišči definicijsko območje funkcije $f(x) = \displaystyle\int_0^1 \frac{dy}{\sqrt{x^2+y^2}}$!

2. Dana je funkcija $F(x) = \displaystyle\int_0^1 \ln(x^2+y^2)\, dy$

   a) Koliko je $F(0)$?

   Poišči naslednje odgovore, ne da bi integral izračunal:

   b) $\lim\limits_{x\to\infty} F(x)$

   c) $F(x)$ je soda funkcija

   d) $F(x)$ je za $x \geq 0$ monotono naraščajoča funkcija

3. Nariši graf funkcije $f(x) = \displaystyle\int_0^1 \operatorname{sgn}(x-y)\, dy$!

4. Izračunaj odvode funkcij:

a) $f(x) = \int\limits_1^2 \frac{\cos(xy)}{y}\, dy$

b) $f(x) = \int\limits_0^x \frac{\ln(1+xy)}{y}\, dy$

c) $f(x) = \int\limits_{x^2}^{x^3} \frac{\sin(x^2 y)}{y}\, dy$

d) $f(x) = \int\limits_x^{x^2} e^{-xy^2}\, dy$

e) $f(x) = \int\limits_{a+x}^{b+x} \frac{\sin(xy)}{y}\, dy$

f) $f(y) = \int\limits_y^{\infty} \frac{e^{-xy}}{x} dx \quad , \quad y > 0$

5. Poišči n–ti odvod funkcije $f(x) = \int\limits_0^x f(t)(x-t)^{n-1}\, dt$!

6. Poišči mešani odvod funkcije $F(x,y) = \int\limits_{\frac{x}{y}}^{xy} (x - yz) f(z)\, dz$!

7. Poišči $F''(x)$ za funkcijo $F(x) = \int\limits_0^x (x+y) f(y)\, dy$!

8. Določi števili a in b tako, da bo imel integral $\int\limits_p^q (f(x) - (a+bx))^2\, dx$

minimalno vrednost:

a) $f(x) = x^2 \quad , \quad p = 1\, ,\, q = 3$

b) $f(x) = \sqrt{1+x^2} \quad , \quad p = 0\, ,\, q = 1$

9. Poišči funkcijo $g(x) = y''(x) + y(x)$, kjer je $y(x) = \int\limits_0^{\infty} \frac{e^{-xz}}{1+z^2}\, dz$!

10. Pokaži, da funkcija $y(x) = \int\limits_{-1}^{1} (z^2-1)^{n-1} e^{xz}\, dz$ ustreza diferencialni enačbi $xy'' + 2ny' - xy = 0$!

11. Poišči ekstrem funkcije:

a) $F(y) = \int\limits_y^{y^2} \frac{dx}{\ln^2 x}$, $y > 1$

b) $F(x) = \int\limits_x^{2x} \frac{e^t}{t^2}\, dt$, $x > 0$

c) $F(x) = \int\limits_{x^2}^{x^2+1} e^{-t^2}\, dt$

12. Pokaži, da je funkcija $F(y) = \int\limits_0^{\frac{\pi}{2y}} \frac{\sin xy}{x}\, dx$ konstanta!

13. Pokaži, da je funkcija $F(y) = \int\limits_{\frac{1}{y}}^{\infty} \frac{\ln(1+y^2x^2)}{x^2}\, dx$ linearna!

14. S pomočjo odvajanja na parameter izračunaj integrale:

a) $\displaystyle\int_0^\infty \frac{\sin x}{x} e^{-xy}\,dx$ $\qquad y > 0$

b) $\displaystyle\int_0^\pi \frac{\ln(1 + a\cos x)}{\cos x}\,dx$ $\qquad |\,a\,| < 1$

c) $\displaystyle\int_0^{\frac{\pi}{2}} \operatorname{arctg}(a \operatorname{tg} x) \operatorname{ctg} x\,dx$ $\qquad a \geq 0$

d) $\displaystyle\int_0^{\frac{\pi}{2}} \ln\frac{1 + a\cos x}{1 - a\cos x} \cdot \frac{dx}{\cos x}$ $\qquad |\,a\,| < 1$

e) $\displaystyle\int_0^\infty \frac{\operatorname{arctg}(ax)}{x(1 + x^2)}\,dx$ $\qquad a \geq 0$

f) $\displaystyle\int_0^\infty \frac{1 - e^{-ax}}{xe^x}\,dx$ $\qquad a > -1$

g) $\displaystyle\int_0^\infty \frac{1 - e^{-ax^2}}{xe^{x^2}}\,dx$ $\qquad a > -1$

h) $\displaystyle\int_0^\infty \frac{e^{-ax^2} - e^{-bx^2}}{x}\,dx$ $\qquad a > 0\,,\ b > 0$

15. Najprej izračunaj prvi integral na elementaren način. S pomočjo odvajanja oziroma integriranja na parameter izračunaj še drugi integral:

a) $\displaystyle\int_a^b x^y\,dy \qquad \int_0^1 \frac{x^b - x^a}{\ln x}\,dx \qquad a, b > 0$

b) $\displaystyle\int_0^\infty \frac{dx}{x^2 + a} \qquad \int_0^\infty \frac{dx}{(x^2 + a)^2} \qquad a > 0$

c) $\displaystyle\int_0^\infty \frac{dx}{x^2 + a} \qquad \int_0^\infty \frac{dx}{(x^2 + a)^3} \qquad a > 0$

d) $\displaystyle\int_a^b e^{-xy}\,dy \qquad \int_0^\infty \frac{e^{-ax} - e^{-bx}}{x}\,dx \qquad a, b > 0$

e) $\displaystyle\int_a^b e^{-xy}\,dy \qquad \int_0^\infty \frac{(e^{-ax} - e^{-bx})}{x}\sin(nx)\,dx \qquad a, b > 0$

f) $\displaystyle\int_0^1 x^y\,dx \qquad \int_0^1 x^y \ln^n x\,dx$

g) $\displaystyle\int_0^y \frac{dx}{x^2 + y^2} \qquad \int_0^y \frac{dx}{(x^2 + y^2)^2}$

h) $\displaystyle\int_0^\infty e^{-ax^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{a}} \qquad \int_0^\infty e^{-ax^2} x^2\,dx \qquad a > 0$

DVOJNI INTEGRAL

polarne koordinate	$x = r\cos\varphi$ $y = r\sin\varphi$ Jacobijeva det.	$0 \leq \varphi \leq 2\pi$ $r \geq 0$ $J(r,\varphi) = r$
ploščina lika$D$	$pl = \iint\limits_D dxdy$	
volumen telesa s streho $z = f(x,y)$ in projekcijo D	$V = \iint\limits_D f(x,y)\, dxdy$	
površina ploskve $z = f(x,y)$ s projekcijo D	$P = \iint\limits_D \sqrt{1 + (\frac{\partial f}{\partial x})^2 + (\frac{\partial f}{\partial y})^2}\, dxdy$	

1. Prevedi dvojni integral $\iint\limits_D f(x,y)\, dxdy$ na dvakratnega:

   a) D je lik omejen s parabolo $y = 2x^2$ in daljico AB, kjer je $A(-1,2)$ in $B(1,2)$

   b) D je tisti del ravnine, ki vsebuje izhodišče in je omejen s hiperbolo $y^2 - x^2 = 1$ in krožnico $x^2 + y^2 = 9$

   c) D je trikotnik s stranicami $y = x$, $y = 2x$, $x + y = 6$

   d) $D:\ x^2 + y^2 \leq 2$, $y \geq x$, $y \geq -x$

   e) $D:\ x^2 + y^2 \leq 1$, $x \geq 0$, $y \geq 0$

   f) $D:\ x + y \leq 1$, $x - y \leq 1$, $x \geq 0$

   g) $D:\ y \geq x^2$, $y \leq 4 - x^2$

   h) $D:\ y - 2x \leq 0$, $2y - x \geq 0$, $xy \leq 2$

2. Zamenjaj vrstni red integracije:

a) $$\int_0^{\pi} dx \int_0^{\sin x} f(x,y)\, dy$$

b) $$\int_0^{4} dx \int_{3x^2}^{12x} f(x,y)\, dy$$

c) $$\int_0^{1} dx \int_{e^{-x}}^{e^x} f(x,y)\, dy$$

d) $$\int_0^{1} dy \int_{-\sqrt{1-y^2}}^{1-y} f(x,y)\, dx$$

e) $$\int_{-1}^{1} dx \int_{x^2-1}^{\cos\frac{\pi x}{2}} f(x,y)\, dy$$

f) $$\int_{-1}^{0} dx \int_{1-\sqrt{-x^2-2x}}^{1-x^2} f(x,y)\, dy$$

g) $$\int_0^{R/\sqrt{2}} dx \int_0^{x} f(x,y)\, dy \;+\; \int_{R/\sqrt{2}}^{R} dx \int_0^{\sqrt{R^2-x^2}} f(x,y)\, dy$$

h) $$\int_{-1}^{0} dy \int_{-2\sqrt{1+y}}^{2\sqrt{1+y}} f(x,y)\, dx \;+\; \int_0^{8} dy \int_{-2\sqrt{1+y}}^{2-y} f(x,y)\, dx$$

3. Izračunaj dvojni integral $\iint\limits_D f(x,y)\,dxdy$:

a) $f = xy$ $\quad D:\ x \geq 0\ ,\ y \leq 2\ ,\ y \geq x^2 + 1$

b) $f = x$ $\quad D:$ med krivuljama $y = 2 - x^2\ ,\ y = 2x - 1$

c) $f = y \ln x$ $\quad D:$ med krivuljami $xy = 1\ ,\ y = \sqrt{x}\ ,\ x = 2$

d) $f = xy$ $\quad D:\ 1 \leq x + y \leq 2\ ,\ x \geq 0\ ,\ y \geq 0$

e) $f = |x|(y - x)$ $\quad D:\ |x| + |y| \leq 1$

f) $f = |x| + |y|$ $\quad D:\ x^2 + y^2 \leq 1 - 2xy\ ,\ xy \geq 0$

g) $f = \frac{x}{x^2+y^2}$ $\quad D:\ y \leq x\ ,\ y \geq \frac{x^2}{2}$

h) $f = y^2 \sin x$ $\quad D:\ 0 \leq x \leq \pi\ ,\ 0 \leq y \leq 1 + \cos x$

i) $f = x^2 \sin^2 y$ $\quad D:\ -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}\ ,\ 0 \leq x \leq 3\cos y$

j) $f = xy^2$ $\quad D:\ x \leq p\ ,\ y^2 \leq 2px\ ,\ (p > 0)$

k) $f = \frac{1}{\sqrt{2a-x}}$ $\quad D:\ (x-a)^2 + (y-a)^2 \leq a^2$

l) $f = x$ $\quad D:\ x + y \geq 2\ ,\ x^2 + (y-1)^2 \leq 1$

m) $f = \sqrt{x^2 - y^2}$ $\quad D:$ trikotnik $A(0,0)\ ,\ B(1,-1)\ ,\ C(1,1)$

n) $f = e^{-\frac{x}{y}}$ $\quad D:\ x \geq 0\ ,\ y \leq 1\ ,\ y^2 \geq x\ ,\ y \geq 0$

o) $f = xy$ $\quad D:\ y \geq 0\ ,\ (x-2)^2 + y^2 \leq 1$

p) $f = x^2 + y^2$ $\quad D:\ a \leq y \leq 3a\ ,\ x \leq y \leq x + a$

q) $f = |x - y|$ $\quad D:\ y^2 \leq 2x\ ,\ 0 \leq x \leq 2\ ,\ y \geq 0$

r) $f = y$ $\quad D:$ območje med osjo x in cikloido
$$\begin{matrix} x = a(t - \sin t) \\ y = a(1 - \cos t) \end{matrix}\quad ,\quad 0 \leq t \leq 2\pi$$

s) $f = \sqrt{xy - y^2}$ $\quad D:$ trapez $A(1,1),\ B(5,1),\ C(10,2),\ D(2,2)$

t) $f = x + 2y$ $\quad D:\ x^2 \leq y \leq \sqrt{x}$

u) $f = \frac{x}{x^2+y^2}$ $D\colon\ x \operatorname{tg} x \le y \le x\ ,\ 0 < x < \frac{\pi}{2}$

v) $f = e^{x+y}$ $D\colon\ x \ge 0\ ,\ e^x \le y \le 2$

w) $f = y^3\sqrt{1-x^2-y^4}$ $D\colon\ x^2 + y^4 \le 1\ ,\ x \ge 0\ ,\ y \ge 0$

4. Zapiši dvojni integral $\iint\limits_D f(x,y)\ dxdy$ kot dvakratnega v polarnih koordinatah, če je podano integracijsko območje D:

a) $D : x^2 + y^2 < 1$

b) $D : x^2 + y^2 < 1\ ,\ x > 0\ ,\ y > 0$

c) $D : x^2 + y^2 < 1\ ,\ x + y > 1$

d) $D : x^2 + y^2 > \sqrt{6}x\ ,\ (x^2 + y^2)^2 < 9(x^2 - y^2)\ ,\ x > 0\ ,\ y > 0$

5. V dvojni integral $\iint\limits_D f(x,y)\ dxdy$ uvedi polarne koordinate:

a) $f = \frac{1}{\sqrt{a^2-x^2-y^2}}$ $D\colon\ x^2 + y^2 \le a^2\ ,\ x \ge 0\ ,\ y \ge 0$

b) $f = x + y$ $D\colon\ x^2 + y^2 \le x + y$

c) $f = \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$ $D\colon\ \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1$

d) $f = \sin\sqrt{x^2 + y^2}$ $D\colon\ \pi^2 \le x^2 + y^2 \le 4\pi^2$

e) $f = |\,x\,|\,(y - x)$ $D\colon\ x^2 + y^2 \le 1\ ,\ y \ge 0$

f) $f = xy$ $D\colon\ x^2 + y^2 \le x$

g) $f = \frac{x}{y}$ $D\colon\ x^2 + y^2 \le y\ ,\ x \ge 0$

h) $f = x^2 + y^2$ $D\colon\ |\,y\,| \le |\,x\,|\ ,\ |\,x\,| \le 1$

i) $f = x^2 + y^2$ $D\colon\ (x^2 + y^2)^2 \le a^2(x^2 - y^2)$

j) $f = x + y$ $\qquad$ $D\colon\ (x^2 + y^2)^2 \leq 2a^2xy$

k) $f = \frac{x}{x^2+y^2}$ $\qquad$ $D\colon\ x^2 + y^2 \leq 2\ ,\ y \leq x^2$
$y \geq 0\ ,\ x \geq 0$

6. Dvojni integral $\displaystyle\iint\limits_D f(x,y)\,dxdy$ izračunaj z vpeljavo novih spremenljivk:

a) $f = \sqrt{xy}$ $\qquad$ $D\colon\ 1 \leq y \leq \sqrt{2}\ ,\ 1 \leq xy \leq 4$

b) $f = y^2$ $\qquad$ $D\colon\ 2x^2 + 3y^2 \leq 8$

c) $f = \sqrt{\sqrt{x} + \sqrt{y}}$ $\qquad$ $D\colon\ \sqrt{x}+\sqrt{y} \leq 1\ ,\ x \geq 0\ ,\ y \geq 0$

7. Izračunaj $\displaystyle\iint\limits_D \frac{xy}{x+y}\,dxdy$ z vpeljavo novih spremenljivk u in v:

$$u = \frac{x}{\sqrt{x+y}}\ ,\ v = \frac{y}{\sqrt{x+y}} \qquad D : x \geq 0\ ,\ y \geq 0\ ,\ y \geq x^2 - x\ ,\ x \geq y^2 - y$$

8. Izračunaj izlimitirane dvojne integrale $\displaystyle\iint\limits_D f(x,y)\,dxdy$:

a) $f = e^{-(x+y)}$ $\qquad$ $D\colon\ 0 \leq x \leq y$

b) $f = \frac{1}{x^4+y^2}$ $\qquad$ $D\colon\ x \geq 1\ ,\ y \geq x^2$

c) $f = \ln(x^2 + y^2)$ $\qquad$ $D\colon\ x^2 + y^2 \leq 1$

d) $f = \frac{1}{x^p y^2}$ $\qquad$ $D\colon\ x \geq 1\ ,\ xy \geq 1$

e) $f = (x^2 + y^2)^{-p}$ $\qquad$ $D\colon\ x^2 + y^2 \geq 1$

f) $f = e^{-(x^2+y^2)}$ $\qquad$ $D\colon$ ravnina xy

g) $f = \frac{1}{\sqrt{x^2+y^2}}$ $\qquad$ $D\colon\ x^2 + y^2 \leq x$

h) $f = \frac{1}{1+x^2+y^2}$ $\quad D\colon\ x \geq 0\ ,\ y \geq 0$

i) $f = e^{-(x+y)^2}$ $\quad D\colon\ x \geq 0\ ,\ y \geq 0$

navodilo: vpelji novi spremenljivki $x = r\cos^2\varphi\ ,\ y = r\sin^2\varphi$

9. Izračunaj ploščine likov omejenih z danimi krivuljami:

a) $y^2 = 4ax\ ,\ x+y = 3a\ ,\ y \geq 0$

b) $y^2 = x\ ,\ y^2 = 8x\ ,\ xy = 1\ ,\ xy = 8$

c) $xy = a^2\ ,\ x+y = \frac{5}{2}a\ ,\ (a > 0)$

d) $y^2 = x^2 - x^4\ ,\ x \geq 0$

e) $y = \ln x\ ,\ y = x-1\ ,\ y = -1$

f) $r = a(1+\cos\varphi)\ ,\ r = a\cos\varphi\ ,\ (a > 0)$

g) $x^2+y^2 = 2x\ ,\ x^2+y^2 = 4x\ ,\ y = x\ ,\ y = 0$

h) $(x^2+y^2)^2 = 8a^2xy$

i) $\sqrt{x}+\sqrt{y} = \sqrt{a}\ ,\ x \geq 0\ ,\ y \geq 0$

j) $r \leq 4(1+\cos\varphi)\ ,\ x \geq 3$

k) $y = \frac{8}{x^2+4}\ ,\ x = 2y\ ,\ x = 0$

l) $(x^2+y^2)^2 = 2(x^2-y^2)\ ,\ x^2+y^2 = 2x$

m) $y^2 = x\ ,\ y^2 = 2x\ ,\ y = x\ ,\ y = 2x$

10. Načrtaj lik v ravnini, katerega ploščina je dana z integralom in izračunaj njegovo ploščino:

a) $\displaystyle\int_{\frac{\pi}{4}}^{\operatorname{arctg} 2} d\varphi \int_0^{\frac{3}{\cos\varphi}} r\ dr$ $\qquad$ b) $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_a^{a(1+\cos\varphi)} r\ dr$

11. Poišči volumen in površino telesa omejenega z valjema $x^2+z^2 = a^2$ in $y^2+z^2 = a^2$!

12. Iz krogle $x^2+y^2+z^2 \leq 4$ je izrezan valj $x^2+y^2 < 1$. Kolikšna je prostornina? Kolikšna je površina na sferi?

13. Izračunaj prostornine teles, ki so omejena z danimi ploskvami:

a) $x^2 + y^2 = R^2$, $z \geq 0$, $z \leq y$

b) $z = e^{-x^2-y^2}$, $z = 0$, $x^2 + y^2 = R^2$

c) $z = x^2 + y^2$, $z = 2(x^2 + y^2)$, $y = x$, $y = x^2$

d) $z = x^2 + y^2$, $z = x + y$

e) $x^2 + z^2 = a^2$, $y \geq 0$, $z \geq 0$, $y \leq x$

f) $y = \sqrt{x}$, $y = 2\sqrt{x}$, $x + z = 6$, $z \geq 0$

g) $2z = x^2 + y^2$, $x^2 + y^2 - z^2 = 1$, $z \geq 0$

h) $z = \sqrt{x^2 + y^2}$, $x^2 + y^2 = 2ax$, $z \geq 0$

i) $z = x^2 + y^2 + 1$, $x = 0$, $y = 0$, $x = 4$, $y = 4$

j) $z = x^2 + y^2$, $x^2 + y^2 = x$, $x^2 + y^2 = 2x$, $z = 0$

k) $x^2 + y^2 + z^2 = 4$, $x^2 + y^2 = 3z$, (manjši del)

14. Izračunaj površine na ploskvah:

a) del ravnine $6x + 3y + 2z = 12$ v prvem oktantu

b) del ploskve $z^2 = 2xy$ nad pravokotnikom

$\{ (x, y) \colon \ x \geq 0 , y \geq 0 , x \leq 3 , y \leq 6 \}$

c) del stožca $z = \sqrt{x^2 + y^2}$, ki ga odreže ravnina $z = \sqrt{2}(\frac{x}{2} + 1)$

d) del ploskve $z = xy$, ki ga izreže valj $x^2 + y^2 = 3$

e) del ploskve $\sqrt{(x^2 + y^2)^3} + 3z = 4$, ki ga odreže ravnina $z = 1$

f) del sfere $x^2 + y^2 + z^2 = R^2$ znotraj valja $x^2 + y^2 = Rx$

g) del ravnine $x + y + z = 1$ določen z $y^2 < x < 1$

h) del valja $x^2 + z^2 = 1$, ki je omejen z $y^2 = 1 - x$

15. Pošči težišče homogene plošče med $y^2 = 2x$ in $y = x$!

TROJNI INTEGRAL

Cilindrične koordinate	$x = r\cos\varphi$	$0 \le \varphi \le 2\pi$
	$y = r\sin\varphi$	$r \ge 0$
Jacobijeva det.$= r$	$z = z$	$-\infty < z < \infty$
Sferične koordinate	$x = r\cos\varphi\cos\vartheta$	$0 \le \varphi \le 2\pi$
	$y = r\sin\varphi\cos\vartheta$	$-\frac{\pi}{2} \le \vartheta \le \frac{\pi}{2}$
Jacobijeva det.$= r^2\cos\vartheta$	$z = r\sin\vartheta$	$r \ge 0$

1. Izračunaj trikratne integrale:

a) $$\int\limits_0^a dx \int\limits_0^x dy \int\limits_0^{xy} x^3y^2z\,dz$$

b) $$\int\limits_0^{e-1} dx \int\limits_0^{e-x-1} dy \int\limits_e^{x+y+e} \frac{\ln(z-x-y)}{(x-e)(x+y-e)}\,dz$$

c) $$\int\limits_0^2 dx \int\limits_0^{2\sqrt{x}} dy \int\limits_0^{\sqrt{2x-y^2/2}} x\,dz$$

d) $$\int\limits_0^1 dx \int\limits_0^{\sqrt{1-x^2}} dy \int\limits_0^{\sqrt{1-x^2-y^2}} xyz\,dz$$

2. Izračunaj trojni integral $\iiint\limits_V f(x,y,z)\,dxdydz$:

a) $f = (x+z)$ $\quad V: 0 \le x \le 3\,,\ 0 \le y \le 2\,,\ 0 \le z \le 1$

b) $f = (x+y+z+1)^{-3}$ $\quad V: x \ge 0\,,\ y \ge 0\,,\ z \ge 0\,,\ x+y+z \le 1$

c) $f = xy$ $\quad V: 0 \le z \le xy\,,\ x+y \le 1\,,\ x \ge 0\,,\ y \ge 0$

d) $f = y\cos(z+x)$ $\quad V: 0 \le y \le \sqrt{x}\,,\ 0 \le z \le \frac{\pi}{2} - x$

e) $f = e^{x+2y+3z}$ $\quad V: x \ge 0\,,\ y \ge 0\,,\ z \ge 0\,,\ x+y+z \le 1$

f) $f = x$ $\quad V: x \ge 0\,,\ 0 \le y \le a\,,\ z \ge 0\,,\ x+z \le a$

3. V trojnem integralu $\iiint\limits_V y\,dxdydz$ je integracijsko območje množica $V:\ 0 < x\,,\ 0 < y < 2-x\,,\ 0 < z < x^2$. V nakazanem vrstnem redu integracije zapiši meje, če integral izrazimo s trikratnim integralom:

a) $\int dx \int dy \int y\,dz$

b) $\int dz \int dx \int y\,dy$

c) Izračunaj integral in primerjaj rezultata pod a) in b)!

4. Trojni integral $\iiint\limits_V f(x,y,z)\,dxdydz$ zapiši kot trikratnega v cilindričnih oziroma sferičnih koordinatah:

a) V je telo v prvem oktantu omejeno z valjem $x^2+y^2 = R^2$ in ravninami $z=0\,,\ z=1\,,\ y=x\,,\ y=\sqrt{3}x$

b) V je del krogle $x^2+y^2+z^2 \le R^2$ v prvem oktantu

c) V je telo omejeno z valjem $x^2+y^2 = 2x$, ravnino $z=0$ in paraboloidom $z = x^2+y^2$

d) V je tisti del krogle $x^2+y^2+z^2 \le R^2$, ki se nahaja znotraj valja $(x^2+y^2)^2 = R^2(x^2-y^2)$ in polprostora $x \ge 0$

5. Integral $\iiint\limits_V \sqrt{x^2+y^2}\,dxdydz$, $V:\ x^2+y^2+z^2<2\ ,\ z>\sqrt{x^2+y^2}$

izračunaj najprej z vpeljavo cilindričnih in nato z vpeljavo sferičnih koordinat !

6. Integrale izračunaj z vpeljavo sferičnih oziroma cilindričnih koordinat:

a) $$\int_0^2 dx \int_0^{\sqrt{2x-x^2}} dy \int_0^a z\sqrt{x^2+y^2}\,dz$$

b) $$\int_{-R}^{R} dx \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} dy \int_0^{\sqrt{R^2-x^2-y^2}} (x^2+y^2)\,dz$$

c) $$\int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \int_0^{\sqrt{1-x^2-y^2}} \sqrt{x^2+y^2+z^2}\,dz$$

d) $$\int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} z^2\,dz$$

7. Izračunaj trojni integral $\iiint\limits_V f(x,y,z)\,dxdydz$:

a) $f=x^2+y^2$ $\qquad V:\ x^2+y^2\le(z+1)^2\ ,\ 0\le z\le 1$

b) $f=x^2$ $\qquad V:\ x^2+y^2\le 2z\ ,\ z\le 2$

c) $f=z$ $\qquad V:\ x^2+y^2+z^2\le 4\ ,\ x^2+y^2\le 3z$

d) $f = \sqrt{x^2 + y^2 + z^2}$ $\quad V:\ x^2 + y^2 + z^2 \leq x$

e) $f = \frac{1}{\sqrt{x^2+y^2+(z-2)^2}}$ $\quad V:\ x^2 + y^2 + z^2 \leq 1$

f) $f = \sqrt{x^2 + y^2 + z^2}$ $\quad V:\ x^2 + y^2 + z^2 \leq z$

g) $f = x^2 + y^2$ $\quad V:\ 1 \leq x^2 + y^2 + z^2 \leq 4$

h) $f = \sqrt{x^2 + y^2}$ $\quad V:\ 0 \leq z \leq 1\ ,\ x^2 + y^2 \leq z^2$

i) $f = z\sqrt{x^2 + y^2}$ $\quad V:\ 0 \leq z \leq 3\ ,\ x^2 + y^2 \leq 2x$

j) $f = 1$ $\quad V:\ z \geq 0\ ,\ x^2+y^2 \leq 2Rx\ ,\ x^2+y^2+z^2 \leq 4R^2$

k) $f = x^2 + y^2 + z^2$ $\quad V:\ x^2 + y^2 + z^2 \leq x + y + z$

l) $f = z^2$ $\quad V:\ x^2 + y^2 + z^2 \leq a^2\ ,\ x^2 + y^2 + z^2 \leq 2az$

8. Izračunaj prostornine teles omejenih z danimi ploskvami:

a) $z = 4 - y^2\ ,\ z = y^2 + 2\ ,\ x = -1\ ,\ x = 2$

b) $z = x^2 + y^2\ ,\ z = x + y$

c) $(x^2 + y^2 + z^2)^2 = a^3 z\ ,\ (a > 0)$

d) $(x - 1)^2 + y^2 = z\ ,\ 2x + z = 2$

e) $z = 6 - x^2 - y^2\ ,\ z = \sqrt{x^2 + y^2}\ ,\ z = 0$

f) $x^2 + y^2 + z^2 \leq R^2\ ,\ x^2 + y^2 \geq R(R - 2z)$

g) $(x^2 + y^2)^2 + z^2 = 1$

h) $2z^2 = x^2 + y^2 + a^2\ ,\ z^2 + a^2 = 2(x^2 + y^2)$

9. Poišči težišče $\frac{1}{8}$ krogle $x^2+y^2+z^2 \leq 1$ v prvem oktantu, če je gostota enaka $\rho = \frac{1}{\sqrt{1-(x^2+y^2+z^2)}}$!

10. Krogla z radijem 1 je nehomogena in ima na površni rdečo piko. Gostota krogle je v vsaki njeni točki enaka oddaljenosti od rdeče pike. Kolikšna je masa krogle ?

11. Izračunaj izlimitirane trojne integrale $\iiint\limits_V f(x,y,z)\,dxdydz$:

a) $f = \frac{xy}{(1+x^2+y^2+z^2)^3}$ $\quad V:\ x \geq 0\ ,\ y \geq 0\ ,\ z \geq 0$

b) $f = \ln(x^2+y^2+z^2)$ $\quad V:\ x^2+y^2+z^2 \leq R^2$

c) $f = e^{-(x+y+z)}$ $\quad V:\ x \geq 0\ ,\ y \geq 0\ ,\ 0 \leq z \leq x+y$

d) $f = \frac{1}{(x^2+y^2+z^2)^3}$ $\quad V:\ x \geq 0\ ,\ y \geq 0\ ,\ z \geq 0\ ,\ x^2+y^2+z^2 \geq 1$

TEORIJA POLJA

$\vec{r} = (x,y,z)$	$\operatorname{div}(P,Q,R) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$
$r = \sqrt{x^2+y^2+z^2}$	
$\operatorname{grad} u = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z})$	$\operatorname{rot}(P,Q,R) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$
$\operatorname{grad} f(r) = \frac{f'(r)}{r}\vec{r}$	
$\frac{\partial u}{\partial l} = \operatorname{grad} u \cdot \frac{\vec{l}}{\lvert\vec{l}\rvert}$	$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$

1. Poišči nivojske ploskve skalarnih polj:

   a) $u = \frac{z}{\sqrt{x^2+y^2+z^2}}$

   b) $u = (\vec{i} \times \vec{r}) \cdot (\vec{j} \times \vec{r})$

   c) $u = \frac{\vec{a}\cdot\vec{r}}{\vec{b}\cdot\vec{r}}$, $\vec{a}$ in $\vec{b}$ sta konstantna vektorja

   d) $u = \arcsin \frac{z}{\sqrt{x^2+y^2}}$

   skiciraj nivojske ploskve $u = 0$, $u = -\frac{\pi}{2}$ in $u = \frac{\pi}{6}$!

2. Skalarno polje $u(\vec{r})$ je razmerje med ploščino paralelograma s stranicama $\vec{r}$ in $\vec{i}$ ter ploščino paralelograma s stranicama $\vec{r}$ in $\vec{k}$. Poišči nivojske ploskve $u = 0$, $u = 1$ in $u = \sqrt{2}$!

3. $u = \sqrt{x^2+y^2+(z+8)^2} + \sqrt{x^2+y^2+(z-8)^2}$. Poišči tisto nivojsko ploskev polja u , ki gre skozi točko $M(9,12,28)$!

4. Prepričaj se, da se nivojske ploskve polj $u = x^2+y^2-z^2$ in $v = xz+yz$ sekajo pod pravim kotom !

5. $u = x^2+2y^2+3z^2+xy+3x-2y-6z$. Poišči $\operatorname{grad} u$ v točkah $A(0,0,0)$ in $B(2,0,1)$! V katerih točkah je $\operatorname{grad} u = 0$?

6. $u = x^3 + y^3 + z^3 - 3xyz$. V katerih točkah prostora je

a) gradient polja pravokoten na os z ?

b) gradient polja enak 0 ?

7. Poišči kot φ med gradientoma polja $u = \frac{x}{x^2+y^2+z^2}$ v točkah $A(1,2,2)$ in $B(-3,1,0)$!

8. Določi polje u , če je $\operatorname{grad} u = (yz-1, xz+1, xy+2)$!

9. $u = \ln \frac{1}{r}$. V katerih točkah prostora je $|\operatorname{grad} u| = 1$?

10. Pokaži:

a) $\operatorname{grad}\,(\vec{a} \cdot \vec{r}) = \vec{a}$

b) $\operatorname{grad}\, r = \frac{\vec{r}}{r}$

c) $\operatorname{grad}\, f(r) = \frac{f'(r)}{r}\vec{r}$

11. Pokaži, da je skalarno polje $u = \ln r^2$ rešitev diferencialne enačbe $u = 2\ln 2 - \ln|\operatorname{grad} u|^2$!

12. Izračunaj $\operatorname{grad} \frac{r}{1+r^2}$ v točki $T(1,-2,2)$!

13. V katerih točkah prostora je gradient polja $u = |\vec{i} \times \vec{r}|$ pravokoten na vektor $\vec{i} + \vec{j} + \vec{k}$?

14. Izračunaj odvod skalarnega polja u v točki T v dani smeri $\vec{l}$:

a) $u = \frac{x}{z}$ $\quad T\ (1,-1,3)$

$\vec{l} = -\vec{i} - \vec{k}$

b) $u = e^{x+y+z}$ $\quad T\ (0,0,0)$

$\vec{l} = \vec{i} + 2\vec{j} - 2\vec{k}$

c) $u = xyz$ $\quad T\ (5,1,2)$

$\vec{l}$ je smer od točke T proti točki $S(9,4,14)$

d) $u = xy^2 + z^3 - xyz$ $\quad T\ (1,1,2)$

$\vec{l}$ oklepa s koordinatnimi osmi kote 60°, 45° in 60°

e) $u = \ln(x+y)$ $\quad T\ (1,2,0)$

$\vec{l}$ je smer tangente na parabolo $y^2 = 4x$, $z = 0$

f) $u = xy - z^2$ $\quad T\ (-9,12,10)$

$\vec{l}$ je smer simetrale kota xOy

15. Poišči točke v prostoru, v katerih je odvod polja $u = \frac{1}{r}$ v smeri vektorja $\vec{l} = \vec{i} + \vec{j} + \sqrt{2}\,\vec{k}$ enak 0 !

16. Poišči odvod polja $u = \operatorname{div}(x^5, y^5, z^5)$ v točki $T(2,2,1)$ v smeri zunanje normale sfere $x^2 + y^2 + z^2 = 9$!

17. Izračunaj divergenco vektorskih polj:

a) $\vec{V} = \frac{1}{x^2+y^2}(x,y,0)$

b) $\vec{V} = (3x^2 + 6xz + 3z^2\,,\ 0\,,\ 0)$

c) $\vec{V} = (y^2 + z^2\,,\ z^2 + x^2\,,\ x^2 + y^2)$

d) $\vec{V} = (x^2yz\,,\ xy^2z\,,\ xyz^2)$

e) $\vec{V} = \frac{1}{\sqrt{x^2+y^2}}(-x,y,z)$ v točki $T(3,4,5)$

18. Izračunaj rotor vektorskih polj:

a) $\vec{V} = (x\,,\ 3xy\,,\ 0)$

b) $\vec{V} = (x\,,\ yz\,,\ -x^2 - z^2)$

c) $\vec{V} = (x^2 + y^2\,,\ y^2 + z^2\,,\ z^2 + x^2)$

d) $\vec{V} = (\frac{y}{z}\,,\ \frac{z}{x}\,,\ \frac{x}{y})$ v točki $T(1,2,-2)$

e) $\vec{V} = (\frac{y}{\sqrt{z}}\,,\ -\frac{x}{\sqrt{z}}\,,\ \sqrt{xy})$ v točki $T(1,1,1)$

f) $\vec{V} = (z^2y\,,\ x^2z\,,\ y^2x)$ v točki $T(-2,3,0)$

19. Katera od naslednjih vektorskih polj so potencialna? Potencialnim poljem poišči potencial:

a) $\vec{V} = (x^2\,,\, y^2\,,\, z^2)$

b) $\vec{V} = (y^2 + 3x^2 z\,,\, 2xy\,,\, x^3)$

c) $\vec{V} = (5x^2 y - 4xy\,,\, 3x^2 - 2y\,,\, 0)$

d) $\vec{V} = (x^2 - y^2\,,\, y^2 - z^2\,,\, z^2 - x^2)$

e) $\vec{V} = (\frac{3x^2y^2}{z} - 2x^3\,,\, \frac{2x^3y}{z}\,,\, -\frac{x^3y^2}{z^2})$

20. Pokaži naslednje enakosti:

a) $\operatorname{div}\vec{r} = 3$

b) $\operatorname{rot}\vec{r} = 0$

c) $\operatorname{div}(\vec{a}\times\vec{r}) = 0$

d) $\operatorname{rot}(\vec{a}\times\vec{r}) = 2\vec{a}$

e) $\operatorname{div}(u\vec{V}) = u\operatorname{div}\vec{V} + \operatorname{grad}u\cdot\vec{V}$

f) $\operatorname{rot}(u\vec{V}) = u\operatorname{rot}\vec{V} + \operatorname{grad}u\times\vec{V}$

21. Izračunaj:

a) $\operatorname{div}\frac{\vec{r}}{r}$

b) $\operatorname{div}(-\frac{z\vec{r}}{r})$

c) $\operatorname{div}(f(r)\vec{r})$

d) $\operatorname{div}((\vec{a}\cdot\vec{r})\,\vec{r})$

e) $\operatorname{div}((\vec{a}\cdot\vec{r})\,\vec{b})$

f) $\operatorname{div}(r(\vec{a}\times\vec{r}))$

g) $\operatorname{rot}(\frac{z\vec{r}}{r})$

h) $\operatorname{rot}(f(r)\vec{r})$

i) $\operatorname{rot}(f(r)\vec{a})$

j) $\operatorname{rot}((\vec{a}\cdot\vec{r})\,\vec{b})$

k) $\operatorname{rot}((\vec{a}\cdot\vec{r})\,\vec{r})$

l) $\operatorname{rot}(r(\vec{a}\times\vec{r}))$

m) $\operatorname{rot}\operatorname{grad}u$

n) $\operatorname{div}(r^2\operatorname{grad}r^2)$

o) $\operatorname{div}\operatorname{rot}\vec{V}$

p) $\Delta f(r)$

q) $\Delta((\vec{a}\cdot\vec{r})\vec{r})$

r) $\Delta(f(r)\vec{r})$

22. Določi funkcijo $f(r)$ tako, da bo $\Delta f(r) = \frac{1}{r}$, če $r \neq 0$ in $f(0) = 0$!

23. Določi $f(r)$ tako, da bo polje $\vec{V} = f(r)\vec{r}$ Laplace-ovo ! To je tako polje, za katerega sta $\operatorname{rot}\vec{V}$ in $\operatorname{div}\vec{V}$ enaka 0.

24. Določi $f(r)$ tako, da bo polje $\frac{f(r)}{r}\vec{r}$ solenoidno !

25. Pokaži, da sta spodnji vektorski polji solenoidni:

a) $\vec{V} = r(\vec{c} \times \vec{r})$

b) $\vec{V} = \frac{2\vec{r}}{r^3}$

26. Reši enačbo $\operatorname{rot}\vec{V} = \vec{r} - 3x\vec{i}$!

27. Pokaži, da sta vektorski polji Laplace-ovi:

a) v dani točki prostora je velikost polja obratno sorazmerna kvadratu oddaljenosti točke od izhodišča, smer polja pa je proti izhodišču koordinatnega sistema

b) v dani točki prostora je velikost polja obratno sorazmerna oddaljenosti točke od osi z, polje pa je usmerjeno pravokotno na os z v smeri proti tej osi

KRIVULJNI IN PLOSKOVNI INTEGRAL

$$\int_C f(x,y,z)\,ds = \int_{t_A}^{t_B} f(x(t),y(t),z(t))\sqrt{x'^2+y'^2+z'^2}\,dt$$

$$\int_C P dx + Q dy + R dz = \pm \int_{t_A}^{t_B} (Px' + Qy' + Rz')\,dt$$

$C : \vec{r} = (x(t), y(t), z(t)) \quad , \quad t_A \le t \le t_B$

$$\iint_S f(x,y,z) dS = \iint_D f(x(u,v),y(u,v),z(u,v))\sqrt{EG-F^2}\,dudv$$

$$\iint_S P dydz + Q dzdx + R dxdy = \iint_S \vec{V}\cdot\vec{\nu}\,dS = \pm \iint_D (\vec{V},\vec{r}_u,\vec{r}_v)\,dudv$$

$S : \vec{r} = (x(u,v), y(u,v), z(u,v)) \quad , \quad (u,v) \in D$

$E = \vec{r}_u \cdot \vec{r}_u \quad F = \vec{r}_u \cdot \vec{r}_v \quad G = \vec{r}_v \cdot \vec{r}_v$

Stokesova formula $\displaystyle \int_{\partial S} \vec{V}\cdot d\vec{r} = \iint_S \operatorname{rot}\vec{V}\cdot\vec{\nu}\,dS$

Gaussova formula $\displaystyle \iint_{\partial V} \vec{V}\cdot\vec{\nu}\,dS = \iiint_V \operatorname{div}\vec{V}\,dxdydz$

Greenova formula $\displaystyle \int_{\partial D} P dx + Q dy = \iint_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big) dxdy$

1. Izračunaj krivuljni integral prve vrste $\int\limits_C f(x,y,z)\,ds$:

a) $f=\frac{y}{x}$ $\quad C:\ y=x^2$ od $(0,0)$ do $(\sqrt{2},2)$ v ravnini $z=0$

b) $f=y^2$ $\quad C:\ y=\sqrt{a^2-x^2}\ ,\ z=0$

c) $f=\sqrt{2y^2+z^2}$ $\quad C:\ x^2+y^2+z^2=2a^2\ ,\ x=y$

d) $f=|\,y\,|$ $\quad C:\ (x^2+y^2)^2=a^2(x^2-y^2)\ ,\ z=0$

e) $f=\sqrt{x^2+y^2}$ $\quad C:\ x^2+y^2=ax\ ,\ z=0$

f) $f=y$ $\quad C:$ lok parabole $y^2=2x$, $z=3$ med točkama $A(0,0,3)$ in $B(4,\sqrt{8},3)$

g) $f=(xz)^2$ $\quad C:\ x^2+y^2+z^2=R^2\ ,\ x^2+y^2=\frac{R^2}{2}\ ,\ z\geq 0$

h) $f=x+y$ $\quad C:\ x^2+y^2+z^2=R^2\ ,\ y=x\ ,\ x\geq 0\ ,\ y\geq 0\ ,\ z\geq 0$

i) $f=\operatorname{arctg}\frac{y}{x}$ $\quad C:$ del Arhimedove spirale $r=2\varphi$ znotraj kroga z radijem $\sqrt{5}$ v ravnini xy

j) $f=2z-\sqrt{x^2+y^2}$ $\quad C:\ x=t\cos t\ ,\ y=t\sin t\ ,\ z=t\ ,\ 0\leq t\leq 2\pi$

k) $f=\sqrt[3]{x^4}+\sqrt[3]{y^4}$ $\quad C:$ astroida $\sqrt[3]{x^2}+\sqrt[3]{y^2}=\sqrt[3]{a^2}$ v ravnini xy

l) $f=xy$ $\quad C:$ rob kvadrata $|\,x\,|+|\,y\,|=a\ ,\ a>0$

2. Kolikšna je masa kardioide $r=2(1+\cos\varphi)$ v ravnini xy, če je gostota na enoto dolžine enaka $\rho=\frac{1}{4}\sqrt{r}$?

3. Kolikšna je masa elipse $x^2+y^2=1$, $z+y=1$, če je gostota na dolžinsko enoto enaka $\rho=z/\sqrt{1+x^2}$?

4. Poišči težišče oboda sferičnega trikotnika
$x^2+y^2+z^2=a^2 \quad , \quad x\geq 0\ ,\ y\geq 0\ ,\ z\geq 0$!

5. Izračunaj krivuljne integrale druge vrste:

a) $\displaystyle\int_C 2xy\,dx + x^2\,dy$

C_1: premica $y = x/2$ med $A(0,0)$ in $B(2,1)$

C_2: parabola $y = x^2/4$ med $A(0,0)$ in $B(2,1)$

b) $\displaystyle\int_C x\,dy$

C: daljica od točke $A(a,0)$ do točke $B(0,b)$

c) $\displaystyle\int_C \cos y\,dx - \sin x\,dy$

C: daljica od točke $A(2,-2)$ do točke $B(-2,2)$

d) $\displaystyle\int_C xy\,dx$

C: lok parabole $x = y^2$ od $A(1,-1)$ do $B(1,1)$

e) $\displaystyle\int_C (x+y)\,dx + (x-y)\,dy$

C: elipsa $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ v nasprotni smeri urinega kazalca

f) $\displaystyle\int_C \frac{(x+y)dx+(y-x)dy}{x^2+y^2}$

C: $x^2 + y^2 = a^2$ v nasprotni smeri urinega kazalca

g) $\displaystyle\int_C \operatorname{arctg}\frac{y}{x}\,dx - dy$

C: sklenjena krivulja od $A(0,0)$ do $B(1,1)$ po paraboli $y = x^2$ in od B do A po premici $x = y$

h) $\displaystyle\int_C (x^2+y^2)\,dy$

C: četverokotnik $A(0,0)$, $B(2,0)$, $C(4,4)$, $D(0,4)$ v smeri $A \to B \to C \to D \to A$

i) $\displaystyle\int_C (y^2-z^2)\,dx + 2yz\,dy - x^2 dz$

C: krivulja $x=t$, $y=t^2$, $z=t^3$, $0 \le t \le 1$ v smeri naraščajočega parametra

j) $\displaystyle\int_C y\,dx + z\,dy + x\,dz$

C: en navoj vijačnice $\vec{r} = (a\cos t, a\sin t, bt)$ v smeri naraščajočega parametra

k) $\displaystyle\int_C (y^2-z^2)\,dx + (z^2-x^2)\,dy + (x^2-y^2)\,dz$

C: trikotnik z oglišči $A(1,0,0)$, $B(0,1,0)$, $C(0,0,1)$ v smeri $A \to B \to C \to A$

l) $\displaystyle\int_C xz^2 dy$

C: presečišče ploskev $x^2+y^2+z^2=a^2$, $x^2+y^2=ax$ $(z \ge 0)$ v nasprotni smeri urinega kazalca gledano iz pozitivne osi x

m) $\displaystyle\int_C xy\,dx + yz\,dy + zx\,dz$

C: polkrožnica $x^2+y^2+z^2=2Rx$, $z=x$, $y>0$ od točke $A(0,0,0)$ do točke $B(R,0,R)$

n) $\displaystyle\int_C r^2 \vec{r} \cdot d\vec{r}$

C: $\vec{r} = (\cos t, \sin t, \frac{t}{\pi})$, $0 \le t \le 2\pi$

o) $\int\limits_C x\,dx - y\,dy$

C: sklenjena krivulja, ki je sestavljena iz četrtine astroide $x = \cos^3 t$, $y = \sin^3 t$ v prvem kvadrantu in odsekov na koordinatnih oseh x in y

p) $\int\limits_C y\,dx - x\,dy + xz\,dz$

C: presek polsfere $x^2 + y^2 + z^2 = 1$, $z > 0$ z ravnino $x + y = 1$ od točke $(1,0,0)$ do točke $(0,1,0)$

6. Prepričaj se, da je integral $\int\limits_A^B P\,dx + Q\,dy$ oziroma $\int\limits_A^B \vec{V}\cdot d\vec{r}$ neodvisen od integracijske poti in ga izračunaj:

a) $P = e^x \cos y$, $Q = -e^x \sin y$ $\qquad A(0,0)$, $B(1,\pi)$

b) $P = 2xy$, $Q = x^2$ $\qquad A(0,0)$, $B(2,1)$

c) $P = \frac{x}{x^2+y^2}$, $Q = \frac{y}{x^2+y^2}$ $\qquad A(3,4)$, $B(5,12)$

d) $P = \frac{1}{y}$, $Q = -\frac{x}{y^2}$ $\qquad A(1,2)$, $B(2,1)$

e) $\vec{V} = (x, y, -z)$ $\qquad A(1,0,-3)$, $B(6,4,8)$

f) $\vec{V} = \sin(x+y+z)\,(1,1,1)$ $\qquad A(0,0,0)$, $B(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3})$

g) $\vec{V} = (yz\,,\ zx\,,\ xy)$ $\qquad A(1,2,3)$, $B(3,2,1)$

7. Izračunaj potencial danega vektorskega polja:

a) $\vec{V} = (\frac{3x^2y^2}{z} - 2x^3\,,\ \frac{2x^3y}{z} + 3y^3\,,\ z^3 - \frac{x^3y^2}{z^2})$

b) $\vec{V} = 2\frac{\vec{r}}{r^4}$

c) $\vec{V} = \frac{1}{1+x^2y^2z^2}\,(yz\,,\ zx\,,\ xy)$

d) $\vec{V} = (\frac{1}{z}\,,\ -\frac{3}{z}\,,\ z + \frac{3y-x}{z^2})$

e) $\vec{V} = (1 - \frac{1}{y} + \frac{y}{z}\,,\ \frac{x}{z} + \frac{x}{y^2}\,,\ -\frac{xy}{z^2})$

f) $dU = 4(x^2 - y^2)(x\,dx - y\,dy)$

g) $dU = (\frac{x-2y}{(y-x)^2} + x)\,dx + (\frac{y}{(y-x)^2} - y^2)\,dy$

8. Naj bo C sklenjena krivulja. Pokaži, da sta integrala enaka 0 za poljubno odvedljivo funkcijo f :

a) $\int\limits_C f(xy)\,(y\,dx + x\,dy)$

b) $\int\limits_C f(r)\,\vec{r}\cdot d\vec{r}$

9. Izračunaj ploskovni integral prve vrste $\iint\limits_S f(x,y,z)\,dS$:

a) $f = x^2 + y^2 + z^2$ S: $x^2 + y^2 = R^2$, $0 \le z \le H$

b) $f = x^2 + y^2$ S: površina telesa $\sqrt{x^2 + y^2} \le z \le 1$

c) $f = \frac{1}{(1+x+y)^2}$ S: $x + y + z = 1$, $x \ge 0$, $y \ge 0$, $z \ge 0$

d) $f = \frac{1}{(1+z)^2}$ S: $x^2 + y^2 + z^2 = 1$, $z \ge 0$

e) $f = x^2 + y^2$ S: $x^2 + y^2 + z^2 = a^2$

f) $f = \sqrt{x^2 + y^2}$ S: $\frac{x^2}{a^2} + \frac{y^2}{a^2} = \frac{z^2}{b^2}$, $0 \le z \le b$

g) $f = x^2 + y^2 + z^2$ S: $|\,x\,| + |\,y\,| + |\,z\,| = a$, $(a > 0)$

h) $f = z$ S: $x^2 + y^2 + z^2 = 1$

i) $f = |\,xyz\,|$ S: del paraboloida $z = x^2 + y^2$, ki ga odreže ravnina $z = 1$

j) $f = xy + yz + zx$ S: del ploskve $z = \sqrt{x^2 + y^2}$ znotraj valja $x^2 + y^2 = 2ax$

k) $f = xy$ S: $z = xy$, $0 \le x \le 1$, $0 \le y \le 1$

l) $f = 3z$ S: $z^2 = x^2 + y^2 + 1$, $1 \le z \le \sqrt{2}$

m) $f = x^2 z$ S: $z^2 = x^2 + y^2$, $1 \le z \le 4$

10. Dana je enačba ploskve. Izračunaj površino tistega dela ploskve, ki je določen z neenačbami:

a) $x^2 + y^2 = x \quad , \quad x^2 + y^2 > z^2 \ , \ z > 0$

b) $y = \sqrt{x^2 + z^2} \quad , \quad x + 2y < 6$

c) $x^2 + y^2 + z^2 = 1 \quad , \quad 0 < x < y < \sqrt{3}x \ , \ z > 0$

11. Izračunaj maso parabolične ploskve $z = \frac{1}{2}(x^2 + y^2)$, $0 \le z \le 1$, če je površinska gostota $\rho(x, y, z) = z$!

12. Polsfera $x^2 + y^2 + z^2 = a^2$, $z \ge 0$ je homogena, tj. površinska gostota je konstantna$= \rho$. Izračunaj njen vztrajnostni moment okrog osi z !

13. Kolikšna je površina tistega dela valja $x^2 + y^2 = x\sqrt{2}$, ki leži znotraj sfere $x^2 + y^2 + z^2 = 2$?

14. Izračunaj ploskovne integrale druge vrste:

a) $$\iint_S z\,dxdy + x\,dzdx + xyz\,dxdy$$

S: četrtina plašča valja v smeri stran od izhodišča
$x^2 + y^2 = 4 \ , \ 0 < x \ , \ 0 < y \ , \ 0 < z < 2$

b) $$\iint_S z\,dxdy$$

S: zunanja stran elipsoida $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

c) $$\iint_S x^2 dydz + y^2 dzdx + z^2 dxdy$$

S: zunanja stran polsfere $x^2 + y^2 + z^2 = a^2 \ , \ z > 0$

d) $\displaystyle\iint_S (y-z)\,dydz + (z-x)\,dzdx + (x-y)\,dxdy$

S: zunanja stran plašča stožca
$x^2+y^2=z^2$, $0<z<H$

e) $\displaystyle\iint_S x\,dydz + y\,dzdx + z\,dxdy$

S: zunanja stran sfere $x^2+y^2+z^2=a^2$

f) $\displaystyle\iint_S xz\,dydz + xy\,dzdx + yz\,dxdy$

S: zunanja stran površine telesa
$x^2+y^2 \le R^2$, $x \ge 0$, $y \ge 0$, $0 \le z \le H$

g) $\displaystyle\iint_S xz\,dydz + x^2y\,dzdx + y^2z\,dxdy$

S: zunanja stran površine telesa
$x^2+y^2 \le 1$, $x \ge 0$, $y \ge 0$, $0 \le z \le x^2+y^2$

h) $\displaystyle\iint_S 3x\,dydz + 3y\,dzdx + z\,dxdy$

S: $z = 9 - x^2 - y^2$, $z \ge 0$

15. Izračunaj krivuljne integrale z uporabo Stokesove formule :

a) $\displaystyle\int_C (y+z)\,dx + (z+x)\,dy + (x+y)\,dz$

C: $x = a\sin^2 t$, $y = a\sin 2t$, $z = a\cos^2 t$, $0 \le t \le \pi$
pomoč: ugotovi lego te elipse v prostoru

b) $\displaystyle\int_C (y-z)\,dx + (z-x)\,dy + (x-y)\,dz$

C: elipsa $x^2 + y^2 = a^2$, $x + z = a$ v nasprotni smeri urinega kazalca, gledano iz pozitivne smeri osi x

c) $\displaystyle\int_C (y^2-z^2)\,dx + (z^2-x^2)\,dy + (x^2-y^2)\,dz$

C: presek površine kocke $0 \le x, y, z \le a$ z ravnino $x + y + z = \frac{3}{2}a$ v smeri urinega kazalca, gledano iz izhodišča

d) $\displaystyle\int_C x\,dx + (x+y)\,dy + (x+y+z)\,dz$

C: $x = a\sin t$, $y = a\cos t$, $z = a(\sin t + \cos t)$
$0 \le t \le 2\pi$

e) $\displaystyle\int_C y^2 dx + z^2 dy + x^2 dz$

C: stranice trikotnika $A(1,0,0)$, $B(0,1,0)$, $C(0,0,1)$ v smeri $A \to B \to C \to A$

16. Ravnine $x = 0$, $y = 0$, $z = 0$, $x+y+z = a$ so mejne ploskve tetraedra. Izračunaj pretok vektorskega polja $\vec{V}$ iz tetraedra v smeri navzven :

a) $\vec{V} = \vec{r}$

b) $\vec{V} = (x^2\,,\, y^2\,,\, z^2)$

17. Izračunaj pretok polja $\vec{V} = (x^2, y^2, z^2)$ v smeri zunanje normale skozi sfero $(x-a)^2 + (y-b)^2 + (z-c)^2 = R^2$!

18. Dana je ploskev z enačbo $z = 1 - \sqrt{x^2 + y^2}$, $0 \le z \le 1$. Izračunaj pretok polja $\vec{V} = \vec{r}$ v smeri proti izhodišču skozi to ploskev !

19. Izračunaj integrale v nalogah 14 d), f) in g) z uporabo Gaussove formule !

20. Uporabi Gaussovo formulo za izračun pretoka polja $\vec{V}$ skozi sklenjeno ploskev S v smeri zunanje normale :

a) $\vec{V} = (x^3, y^3, z^3)$ $\quad S\colon\ x^2 + y^2 + z^2 = a^2$

b) $\vec{V} = (x^2, y^2, z^2)$ $\quad S\colon\ \sqrt{x^2 + y^2} \leq z \leq H$

c) $\vec{V} = (x^2, y^2, z^2)$ $\quad S\colon$ površina kocke $0 \leq x, y, z \leq a$

d) $\vec{V} = r^2 \vec{r}$ $\quad S\colon\ x^2 + y^2 + z^2 = 1$

e) $\vec{V} = (xz^2, yx^2, zy^2)$ $\quad S\colon$ površina telesa $x^2 + y^2 + z^2 \leq 2z$, $x^2 + y^2 \geq z^2$

21. Izračunaj pretok polja $\vec{V}$ skozi dano ploskev S ! Navodilo: uporabi Gaussovo formulo za primerno izbrano zaključeno ploskev !

a) $\vec{V} = (z^2, xz, y^2)$ $\quad S : z = 4 - (x^2 + y^2)\ ,\ z > 0$

b) $\vec{V} = \vec{r}$ $\quad S :$ trikotnik $A(1,0,0)$, $B(0,1,0)$, $C(0,0,1)$

c) $\vec{V} = \vec{i} \times \vec{r}$ $\quad S : z = 1 - \sqrt{x^2 + y^2}\ ,\ 0 < z < 1$

d) $\vec{V} = \vec{r}$ $\quad S : |\,x\,| + |\,y\,| + |\,z\,| = 1$

e) $\vec{V} = (1, 2, 3)$ $\quad S : (x-1)^2 + (y-2)^2 + (z-3)^2 = 1\ ,\ x+y+z > 6$

22. Težja naloga. Izračunaj pretok polja $\vec{V} = \frac{\vec{r}}{r^3}$ skozi dano ploskev S ! Navodilo: uporabi Gaussovo formulo za telo, ki ima z majhno kroglico izrezano koordinatno izhodišče, kjer polje $\vec{V}$ ni definirano !

a) $S\colon$ površina kocke $[-a, a]^3$

b) $S\colon\ x^2 + y^2 = (z-1)^2\ ,\ 0 < z < 1$

23. Dano je vektorsko polje $\vec{V} = (x, y, 0)$. Kolikšen je njegov pretok skozi paraboloid $z = 4 - (x^2 + y^2)\ ,\ z > 0$ v smeri navzgor ?

a) Izračunaj pretok direktno s ploskovnim integralom

b) Uporabi Gaussovo formulo, saj je pretok skozi osnovni krog enak 0

24. Uporabi Gaussovo formulo za integral $\displaystyle\iint_S \operatorname{rot} \vec{V} \cdot \vec{\nu}\, dS$!

25. Dokaži, da je pretok konstantnega vektorskega polja $\vec{V} = \vec{c}$ skozi poljubno zaključeno ploskev enak 0!

26. Z uporabo Greenove formule izračunaj integral

$$\int_C (1-x^2)y\,dx + x(1+y^2)\,dy \quad , \quad C\colon\ x^2+y^2=R^2 \text{ v pozitivni smeri !}$$

27. C je sklenjena krivulja simetrična glede na izhodišče. S pomočjo Greenove formule dokaži, da je integral

$$\int_C (yx^3+e^y)\,dx + (xy^3+xe^y-2y)\,dy$$

enak 0 !

28. $$I_1 = \int_{AmB} (x+y)^2dx - (x-y)^2dy \qquad I_2 = \int_{AnB} (x+y)^2dx - (x-y)^2dy$$

Izračunaj $I_1 - I_2$, če je AmB daljica od $A(0,0)$ do $B(1,1)$ in AnB del parabole $y = x^2$ od A do B ! Uporabi Greenovo formulo !

FUNKCIJE KOMPLEKSNE SPREMENLJIVKE

$z = x + iy \quad , \quad \overline{z} = x - iy \qquad w = f(z) = u(x,y) + iv(x,y)$

$e^z = e^x(\cos y + i \sin y) \qquad \ln z = \ln|z| + i\,(\arg z + 2n\pi)$

$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}$

$\operatorname{sh} z = \frac{e^z - e^{-z}}{2} \qquad \operatorname{ch} z = \frac{e^z + e^{-z}}{2}$

Cauchy − Riemannovi enačbi : $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad , \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$

Cauchyjeva formula : $f(z_0) = \frac{1}{2\pi i} \int\limits_{C+} \frac{f(z)}{z - z_0}\, dz$

Izrek o residuih : $\int\limits_{C+} f(z)\, dz = 2\pi i \sum\limits_{z_i \in C} \operatorname*{res}\limits_{z = z_i} f(z)$

$$\int\limits_{-\infty}^{\infty} f(x)\, dx = 2\pi i \sum\limits_{Im(z_i) > 0} \operatorname*{res}\limits_{z = z_i} f(z)$$

Residuum, pol stopnje n : $\operatorname*{res}\limits_{z = z_0} f(z) = \frac{1}{(n-1)!} \lim\limits_{z \to z_0} \left(f(z)(z - z_0)^n\right)^{(n-1)}$

$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots + \frac{z^n}{n!} + \cdots \qquad |z| < \infty$

$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots + (-1)^n \frac{z^{2n+1}}{(2n+1)!} + \cdots \qquad |z| < \infty$

$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots + (-1)^n \frac{z^{2n}}{(2n)!} + \cdots \qquad |z| < \infty$

$\ln(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots + (-1)^{n+1} \frac{z^n}{n} + \cdots \qquad |z| < 1$

$(1 + z)^\alpha = \sum\limits_{n=0}^{\infty} \binom{\alpha}{n} z^n \qquad |z| < 1$

ANALITIČNE FUNKCIJE

1. Načrtaj krivulje v kompleksni ravnini:

   a) $z = 1 - it$ $\qquad 0 \le t \le 2$

   b) $z = \cos t + i \sin t$ $\qquad \frac{\pi}{2} \le t \le \frac{3\pi}{2}$

   c) $z = t + \frac{i}{t}$ $\qquad -\infty < t \le 0$

   d) $z = -t + i\sqrt{1 - t^2}$ $\qquad -1 \le t \le 0$

   e) $z = i(t + i)^2$ $\qquad -1 \le t \le 2$

   f) $z = 1 + \frac{i-t}{i+t}$ $\qquad -\infty < t < \infty$

   g) $\{ z \,, |z| = \mathrm{Re}\,(z + 1) \}$

   h) $\{ z \,, |\frac{z-i}{z+i}| = \sqrt{2} \}$

2. Načrtaj krivulje za vrednosti konstante $C = -1\,,\, 0\,,\, \frac{1}{2}\,,\, 2$:

   a) $\{ z \,, \mathrm{Re}(\frac{1}{z}) = C \}$

   b) $\{ z \,, \mathrm{Re}(z^2) = C \}$

   c) $\{ z \,, \mathrm{Im}(z^2) = C \}$

   d) $\{ z \,, \mathrm{Im}(\frac{1}{z}) = C \}$

3. Poišči množice v kompleksni ravnini:

   a) $\{ z \,, 1 < \mathrm{Re}(z) < 2 \}$

   b) $\{ z \,, 1 \le |z + i| \le 2 \}$

   c) $\{ z \,, |z - 1| < |z - i| \}$

   d) $\{ z \,, |z| > |\sqrt{3} + i\mathrm{Im}(z + i)\,| \}$

   e) $\{ z \,, \mathrm{Im}(\overline{z^2}) < -1 \}$

   f) $\{ z \,, z^2 + \overline{z^2} = 1 \}$

   g) $\{ z \,, z\overline{z} - (z + \overline{z}) - 2 = 0 \}$

   h) $\{ z \,, \frac{\pi}{4} \le \arg z \le \frac{\pi}{2} \text{ in } \mathrm{Im}(z) \le 1 \}$

4. Funkcijam poišči realni in imaginarni del:

   a) $w = z^3 - i\overline{z}$

   b) $w = \frac{\overline{z}}{z}$

   c) $w = \frac{|z|^2}{z}$

5. Poišči enačbo krivulje v ravnini $w=u+iv$, v katero preslika funkcija $w=f(z)$ dano krivuljo ravnine $z=x+iy$:

a)	$w=z^2$ premica $x=1$		f)	$w=\frac{1}{z}$	$x^2+y^2=Cx$
b)	$w=z^2$ premica $y=\frac{1}{2}$		g)	$w=\frac{1}{z}$	$y=Cx$
c)	$w=z^2$ premica $x=y$		h)	$w=\frac{1}{z}$	$\arg(z^2)=-\frac{\pi}{2}$
d)	$w=\frac{1}{z}$ premica $x=C$		i)	$w=z^2+z$	$x=-1$
e)	$w=\frac{1}{z}$ premica $y=C$				

6. Katere izmed naslednjih funkcij so analitične :

a) $f(z)=\frac{i}{z}$ c) $f(z)=|z|$

b) $f(z)=\frac{\mathrm{Re}z}{\mathrm{Im}z}$ d) $f(z)=z^3$

7. Katere od funkcij so harmonične :

a) $u=2e^x\cos y$ c) $u=\mathrm{arctg}(xy)$

b) $u=x^2+2x-y^2$ d) $u=x\ln y$

8. Ali sta funkciji u in v konjugirana para harmoničnih funkcij ?

a) $u=3(x^2-y^2)$ $v=3x^2y-y^3$

b) $u=\frac{x}{x^2+y^2}$ $v=\frac{-y}{x^2+y^2}$

c) $u=x$ $v=-y$

d) $u=e^x\cos y+1$ $v=e^x\sin y+1$

9. Ali obstaja analitična funkcija $f=u+iv$, za katero je :

a) $u=e^{\frac{y}{x}}$ c) $v=\ln(x^2+y^2)-x^2+y^2$

b) $u=\frac{x^2-y^2}{(x^2+y^2)^2}$ d) $u=x^2$

10. Poišči konjugirano harmonično funkcijo :

a) $u=x^2-y^2+x$ c) $v=\mathrm{arctg}\,\frac{y}{x}$

b) $u=\frac{x}{x^2+y^2}$ d) $u=xy$

11. Poišči $f(z)$, če je $f(z) = u + iv$ analitična funkcija:

a) $u = x^2 - y^2 + 5x + y - \frac{y}{x^2+y^2}$

b) $v = 3 + x^2 - y^2 - \frac{y}{2(x^2+y^2)}$

c) $v = \ln(x^2 + y^2) + x - 2y$

d) $v = 2x(1 - y)$

e) $u = e^{x^2-y^2} \cos(2xy)$ $\qquad f(0) = 1$

f) $u = e^{-x}(x \cos y + y \sin y)$ $\qquad f(0) = i$

g) $v = \frac{x+y}{x^2+y^2}$ $\qquad f(i) = 1 + i$

h) $v = 3x + 2xy$ $\qquad f(-i) = 2$

12. Določi realne koeficiente a, b in c tako, da bo funkcija $f(z) = x + ay + i(bx + cy)$ analitična !

13. $u(x, y)$ je harmonična funkcija.

a) Ali je $f(x, y) = u^2(x, y)$ harmonična funkcija?

b) Kakšna mora biti funkcija $f(u)$, da bo $f(u(x, y))$ harmonična funkcija?

14. Poišči funkcijo f, za katero je $u(x, y)$ harmonična funkcija:

a) $u = f(\frac{y}{x})$

b) $u = f(x^2 + y^2)$

15. Poišči vse analitične funkcije, katerih realni del je oblike $g(x) + h(y)$!

16. Funkcija $f(z) = u(x, y) + iv(x, y)$ je analitična. Pokaži, da se krivulji $u(x, y) = a$ in $v(x, y) = b$ sekata pod pravim kotom !

ELEMENTARNE FUNKCIJE

1. Izračunaj:

a) $e^{\frac{2-\pi i}{4}}$

b) $\operatorname{ctg}(\frac{\pi}{4} - i\ln 2)$

c) $\operatorname{th}(\ln 3 + \frac{\pi}{4}i)$

d) $\ln\frac{1+i}{\sqrt{2}}$

e) $\ln(2-3i)$

f) 1^{-i}

g) $\left(\frac{1-i}{\sqrt{2}}\right)^{1+i}$

h) $\arcsin i$

i) $e^{\ln 7 + \pi i}$

j) $\sin(-1+2i)$

k) i^i

l) $|(-i)^{-i}|$

m) $\left|\operatorname{th}\frac{\pi(1+i)}{4}\right|$

n) $\cos(\frac{\pi}{2} + i\ln 2)$

o) $\left|\sin(\pi + i\ln(2+\sqrt{5}))\right|$

p) $(-1)^{\sqrt{2}}$

q) 1^i

r) $\ln(-1-i)$

s) $\operatorname{tg}\frac{\pi i}{2}$

t) $e^{e^{(1+\frac{\pi i}{2})}}$

2. z, u in v so poljubna kompleksna števila. Dokaži:

a) $e^u e^v = e^{u+v}$

b) $e^{z+2\pi i} = e^z$

c) $\sin^2 z + \cos^2 z = 1$

d) $\sin z = \cos(\frac{\pi}{2} - z)$

e) $\sin(u+v) = \sin u \cos v + \cos u \sin v$

f) $\sin(iz) = i \operatorname{sh} z$

g) $\cos(iz) = \operatorname{ch} z$

h) $|\sin z| = \sqrt{\sin^2 x + \operatorname{sh}^2 y}$

i) $\overline{\sin z} = \sin \overline{z}$

j) $\cos 2z = \cos^2 z - \sin^2 z$

3. Poišči realni in imaginarni del funkcij $\sin z$, $\cos z$ in $\operatorname{ch}(z-i)$!

4. Za katera kompleksna števila z je vrednost funkcije $f(z)$ realno število? Za katere z je $f(z)$ čisto imaginarno število?

a) $f(z) = e^z$

b) $f(z) = \sin z$

c) $f(z) = \operatorname{ch} z$

d) $f(z) = \cos z$

5. Prepričaj se, da imata kompleksni enačbi

$\sin z = 0$

$\cos z = 0$

natanko iste rešitve kot ustrezni realni enačbi!

6. Reši enačbe:

a) $|\operatorname{tg} z| = 1$

b) $\sin z + \cos z = 2$

c) $\sin z - \cos z = i$

d) $\operatorname{sh} z - \operatorname{ch} z = 2i$

e) $\operatorname{sh}(iz) = -i$

f) $e^{2z} + 2e^z - 3 = 0$

g) $\cos z = 2$

h) $4\cos z + 5 = 0$

i) $\ln(i - z) = 1$

j) $\sin z = 3$

k) $e^z + i = 0$

l) $|e^{-2iz}| = 4$

m) $e^{z^2} = 1$

INTEGRAL

1. Izračunaj $\int\limits_C f(z)\,dz$ po krivulji C od točke z_1 do točke z_2:

a) $f(z) = 1 + i - 2\overline{z}$ $\qquad z_1 = 0\,,\; z_2 = 1 + i$

C_1: premica $y = x$

C_2: parabola $y = x^2$

C_3: lomljena linija $z_1 \to z_3 \to z_2\,,\; z_3 = 1$

b) $f(z) = z^2 + z\overline{z}$ $\qquad z_1 = 1\,,\; z_2 = -1$

C: polkrožnica $|z| = 1\,,\; \mathrm{Im} z > 0$

c) $f(z) = e^{\overline{z}}$ $\qquad z_1 = 0\,,\; z_2 = \pi - i\pi$

C: premica $y = -x$

d) $f(z) = z\mathrm{Im}(z^2)$ $\qquad z_1 = 1 - i\,,\; z_2 = 1 + i$

C: daljica $z_1 \to z_2$

e) $f(z) = |z|$

C_1: daljica $z_1 = 0 \to z_2 = 2 - i$

C_2: polkrožnica $|z| = 2\,, \mathrm{Re} z > 0\,, z_1 = -2i \to z_2 = 2i$

2. Izračunaj integral analitične funkcije $\int\limits_{z_1}^{z_2} f(z)\,dz$:

a) $f(z) = 3z^2 + 2z$ $\qquad z_1 = 1 - i \quad z_2 = 2 + i$

b) $f(z) = z\cos z$ $\qquad z_1 = 0 \quad z_2 = i$

c) $f(z) = ze^z$ $\qquad z_1 = 1 \quad z_2 = i$

d) $f(z) = (z - i)e^{-z}$ $\qquad z_1 = 0 \quad z_2 = i$

e) $f(z) = (z^3 - z)e^{z^2/2}$ $\qquad z_1 = 1 + i \quad z_2 = 2i$

f) $f(z) = \cos z$ $\quad z_1 = \frac{\pi}{2} \quad z_2 = \pi + i$

g) $f(z) = 2z + 1$ $\quad z_1 = 1+i \quad z_2 = -1 - i$

h) $f(z) = ze^{z^2}$ $\quad z_1 = -i \quad z_2 = i$

i) $f(z) = \sin z \cos z$ $\quad z_1 = 0 \quad z_2 = 1 + i$

j) $f(z) = \frac{1+\mathrm{tg}\, z}{\cos^2 z}$ $\quad z_1 = 1 \quad z_2 = i$

k) $f(z) = \frac{1}{\sqrt{z}}$, $\sqrt{1}=-1$ $\quad z_1 = 1 \quad z_2 = -1 \quad \mathrm{Im} z > 0$

l) $f(z) = \frac{\ln^3 z}{z}$ $\quad z_1 = 1 \quad z_2 = i \quad \mathrm{Im} z > 0$

glavna veja logaritma

m) $f(z) = \frac{\ln z}{z}$ $\quad z_1 = 1 \quad z_2 = i \quad \mathrm{Im} z > 0$

glavna veja logaritma

3. Izračunaj integral $\int\limits_C f(z)\, dz$ z uporabo Cauchyjeve formule! Smer integracije je nasprotna smeri urinega kazalca.

a) $f(z) = \dfrac{e^{z^2}}{z^2 - 6z}$ $\quad C:\ |z - 2| = 1$

b) $f(z) = \dfrac{e^{z^2}}{z^2 - 6z}$ $\quad C:\ |z - 2| = 3$

c) $f(z) = \dfrac{e^{z}}{z^2 + 2z}$ $\quad C:\ |z| = 1$

d) $f(z) = \dfrac{e^{iz}}{z^2 + 1}$ $\quad C:\ |z - i| = 1$

e) $f(z) = \dfrac{\sin \frac{\pi z}{2}}{z^2 + 2z - 3}$ $\quad C:\ |z - i| = 2$

f) $f(z) = \dfrac{\sin(iz)}{z^2 - 4z + 3}$ $\quad C:\ |z| = 2$

g) $f(z) = \dfrac{e^{2z}}{z^3 - 1}$ $\quad C:\ x^2 + y^2 = 2x$

LAURENT-OVA VRSTA

1. Poišči konvergenčno območje vrste $\sum\limits_{n=1}^{\infty} a_n$:

a) $a_n = (1+i)^n z^n$

b) $a_n = e^{in} z^n$

c) $a_n = (\frac{z}{1-i})^n$

d) $a_n = (\frac{iz}{n})^n$

e) $a_n = (3+i^n)(\frac{z}{2})^n$

f) $a_n = \cos(in) z^n$

g) $a_n = \frac{(1+i)^{n+1}}{z^n}$

h) $a_n = \frac{\sin(in)}{(z+i)^n}$

i) $a_n = e^n (iz)^{-n}$

j) $a_n = \frac{n 2^{-n}}{(z-2-i)^n}$

k) $a_n = \frac{3^n+2}{(z+2i)^n}$

l) $a_n = (in + \frac{1}{n})(z+1+i)^n$

m) $a_n = (n+2i) z^n$

n) $a_n = n!(iz)^n$

2. Poišči konvergenčno območje vrst:

a) $\sum\limits_{n=0}^{\infty} (\frac{z+2i}{6})^n + \sum\limits_{n=1}^{\infty} (\frac{3+4i}{z+2i})^n$

b) $\sum\limits_{n=0}^{\infty} (1+in)\,(z-2+i)^n + \sum\limits_{n=1}^{\infty} (\frac{1}{n(z-2+i)})^n$

c) $\sum\limits_{n=0}^{\infty} (\frac{z}{4})^n + \sum\limits_{n=1}^{\infty} (\frac{2}{z})^n$

d) $\sum\limits_{n=0}^{\infty} n(z+1-i)^n + \sum\limits_{n=1}^{\infty} \frac{n}{(z+1-i)^n}$

e) $\sum\limits_{n=0}^{\infty} \frac{z^n}{2^{n+1}} + \sum\limits_{n=1}^{\infty} \frac{1}{z^n}$

f) $\sum\limits_{n=0}^{\infty} (\frac{z+1}{n+i})^n + \sum\limits_{n=1}^{\infty} \frac{2^n-1}{(z+1)^n}$

g) $\frac{1}{z} + \sum\limits_{n=0}^{\infty} z^n$

3. Funkcijo $f(z)$ razvij v Taylorjevo vrsto okrog izhodišča in poišči konvergenčni radij:

a) $f(z) = \frac{1}{z-2}$

b) $f(z) = \frac{z}{z^2-2z-3}$

c) $f(z) = \frac{z+1}{z^2+4z-5}$

d) $f(z) = \frac{z}{z^2+i}$

e) $f(z) = \cos^2 \frac{iz}{2}$

f) $f(z) = \ln \sqrt{4-2z}$

g) $f(z) = e^z \sin z$

h) $f(z) = \operatorname{tg} z$

4. Zapiši Taylorjevo vrsto funkcije $f(z)$ v okolici točke z_0 in določi konvergenčni radij:

a) $f(z) = \frac{1}{1-z}$ $\quad z_0 = 3i$

b) $f(z) = \sin(2z+1)$ $\quad z_0 = -1$

c) $f(z) = e^z$ $\quad z_0 = \pi i$

d) $f(z) = z \ln z$ $\quad z_0 = 1$

V nalogah 5,6 in 7 dano funkcijo razstavi na parcialne ulomke in vsak ulomek razvij v vrsto z uporabo geometrijske vrste.

5. Funkcijo $f(z)$ razvij v Laurentovo vrsto v okolici vsake njene singularne točke in določi območje konvergence:

a) $f(z) = \frac{1}{z(z-1)}$

b) $f(z) = \frac{1}{z^2+z}$

c) $f(z) = \frac{1}{z^2+1}$

d) $f(z) = \frac{2z-3}{z^2-3z+2}$

6. Poišči Laurentovo vrsto funkcije $f(z)$ na danem kolobarju:

a) $f(z) = \dfrac{1}{z^2+z}$ $\quad 1 < |z| < \infty$

b) $f(z) = \dfrac{1}{z(z-1)}$ $\quad 1 < |z-1| < \infty$

c) $f(z) = \dfrac{2z+3}{z^2+3z+2}$ $\quad 1 < |z| < 2$

d) $f(z) = \dfrac{2}{(z^2-4)(z^2-1)}$ $\quad 1 < |z| < 2$

e) $f(z) = \dfrac{2}{z^2-1}$ $\quad 1 < |z+2| < 3$

7. Razvij funkcijo $f(z)$ po potencah z-ja. Vrsta naj bo konvergentna na predpisanem kolobarju:

a) $f(z) = \dfrac{1}{(z-2)(z-3)}$ $\quad |z| < 2$

b) $f(z) = \dfrac{1}{(z-2)(z-3)}$ $\quad 2 < |z| < 3$

c) $f(z) = \dfrac{1}{(z-2)(z-3)}$ $\quad 3 < |z| < \infty$

d) $f(z) = \dfrac{2z+1}{z^2+z-2}$ $\quad |z| < 1$

e) $f(z) = \dfrac{2z+1}{z^2+z-2}$ $\quad 1 < |z| < 2$

f) $f(z) = \dfrac{2z+1}{z^2+z-2}$ $\quad 2 < |z| < \infty$

8. Uporabi binomsko formulo in zapiši nekaj členov Laurentove vrste za funkcijo $f(z) = \frac{1}{(z^2-1)^2}$ v okolici točke $z_0 = 1$!

9. Poišči Laurentovo vrsto funkcije $f(z)$ v okolici točke z_0:

a) $f(z) = \frac{\sin z}{z-\pi}$ $\quad z_0 = \pi$

b) $f(z) = \frac{z}{(z+1)^2}$ $\quad z_0 = -1$

c) $f(z) = ze^{\frac{1}{z+i}}$ $\quad z_0 = -i$

d) $f(z) = \frac{z-2iz-ie^{\frac{1}{z-i}}}{z-i}$ $\quad z_0 = i$

10. Zapiši nekaj členov Laurentove vrste za funkcijo $f(z)$ v okolici izhodišča:

a) $f(z) = \frac{\sin z}{z^2}$

b) $f(z) = \frac{\sin^2 z}{z}$

c) $f(z) = \frac{e^z}{z^3}$

d) $f(z) = z^3 e^{\frac{1}{z}}$

e) $f(z) = \frac{1+\cos z}{z^4}$

f) $f(z) = \frac{1-e^{-z}}{z^3}$

g) $f(z) = \frac{1}{z^2(1-z)}$

h) $f(z) = z^2 \cos \frac{1}{z}$

SINGULARNE TOČKE IN UPORABA

1. Poišči ničle funkcije $f(z)$ in njihove stopnje:

a) $f(z) = z^4 + 4z^2$

b) $f(z) = \sin z^2$

c) $f(z) = z^2 \sin z$

d) $f(z) = (1 + \cos z)^3$

e) $f(z) = 1 - e^z$

f) $f(z) = (z^2 + \pi^2)\,(1 + e^{-z})$

g) $f(z) = (z^2 - 1)\ln^2 z$, glavna veja logaritma

2. $z = 0$ je ničla funkcije $f(z)$. Določi njeno stopnjo:

a) $f(z) = z^2(e^{z^2} - 1)$

b) $f(z) = (e^z - e^{z^2})\ln(1 - z)$

c) $f(z) = 6 \sin z^3 + z^3(z^6 - 6)$

d) $f(z) = e^{\sin z} - \cos z$

3. Funkcijam poišči singularne točke in tip singularnosti:

a) $f(z) = \dfrac{\sin z}{z^3 + z^2 - z - 1}$

b) $f(z) = \dfrac{1+z}{z^3}$

c) $f(z) = e^{\frac{1}{z^2}}$

d) $f(z) = \dfrac{2z-1}{1-\sin(\pi z)}$

e) $f(z) = e^{\frac{1}{z+2}}$

f) $f(z) = \dfrac{z}{z^5 - 2z^4 + 2z^3}$

g) $f(z) = \dfrac{(z-i)e^{\frac{1}{z-i}}}{z^2+1}$

h) $f(z) = \dfrac{z^2-1}{z^6 + 2z^5 + z^4}$

i) $f(z) = \dfrac{\operatorname{tg} z}{z}$

4. $z = 0$ je singularna točka funkcije $f(z)$. Kakšnega tipa je?

a) $f(z) = \dfrac{z^6}{z - \sin z}$

b) $f(z) = \dfrac{\operatorname{sh} z}{z^4}$

c) $f(z) = \dfrac{z^3}{1 - \cos z^2}$

d) $f(z) = \dfrac{z - \sin z}{(1-\cos 2z)^2}$

e) $f(z) = \dfrac{3}{2 + z^2 - e^z - e^{-z}}$

f) $f(z) = \dfrac{z \sin z}{e^{-z} + z - 1}$

g) $f(z) = \dfrac{1}{e^{-z}-1} + \dfrac{1}{z^2}$

h) $f(z) = \dfrac{z}{(e^{-z}-1)^3} + \dfrac{1}{z}$

i) $f(z) = e^{\frac{1}{z}} \operatorname{ch} z$

5. Kakšna singularnost je število z_0 za funkcijo $f(z)$?

a) $f(z) = \dfrac{\sin(\pi z)}{2e^{z-1} - z^2 - 1}$ $\qquad z_0 = 1$

b) $f(z) = \dfrac{z^3 - z^2 + z - 6}{\sin^2(\pi z)}$ $\qquad z_0 = 2$

6. Kaj je točka ∞ za funkcijo $f(z)$?

a) $f(z) = \dfrac{z^2}{5 - 2z^2}$

b) $f(z) = \dfrac{3z^2 - 5z + 2}{z^2 + z - 4}$

c) $f(z) = \dfrac{z}{1 - 3z^3}$

d) $f(z) = 1 - z + 2z^2$

e) $f(z) = e^{-z^2}$

f) $f(z) = e^{\frac{1}{z}} + 2z^2 - 5$

7. V singularnih točkah izračunaj residuume:

a) $f(z) = \dfrac{z}{(z+1)^3(z-2)^2}$

b) $f(z) = \dfrac{z^2 + 1}{z - 2}$

c) $f(z) = \dfrac{z^2}{(z^2+1)^2}$

d) $f(z) = \dfrac{e^{\pi z}}{z - i}$

e) $f(z) = \dfrac{e^{iz}}{z^2 + 9}$

f) $f(z) = \operatorname{tg} z$

g) $f(z) = \dfrac{\cos z}{z^3 - \frac{\pi z^2}{2}}$

h) $f(z) = \dfrac{\operatorname{ch} z}{(z^2+1)(z-3)}$

i) $f(z) = z^3 \sin \dfrac{1}{z^2}$

j) $f(z) = (z-1)^2 \sin \dfrac{1}{z}$

k) $f(z) = \dfrac{e^{\frac{1}{z}}}{1 - z}$

8. Izračunaj residuum funkcije $\dfrac{z^3}{1 - \cos z^2}$ v točki $z = 0$!

9. Izračunaj integral $\int\limits_C f(z)\,dz$ z uporabo izreka o residuih! Smer integracije je nasprotna smeri urinega kazalca:

a) $f(z) = \dfrac{1}{z^2+16}$ $\qquad C: \ |z| = 5$

b) $f(z) = \dfrac{z^2}{(z^2+1)(z-2)}$ $\qquad C: \ |z| = 8$

c) $f(z) = \dfrac{z}{(z-1)^2(z+2)}$ $\qquad C: \ x^{\frac{2}{3}} + y^{\frac{2}{3}} = 3^{\frac{2}{3}}$

d) $f(z) = \dfrac{e^{z^2}-1}{z^3-iz^2}$ $\qquad C: \ |z-i| = 3$

e) $f(z) = \dfrac{z}{e^z+3}$ $\qquad C: \ |z| = 6$

f) $f(z) = z^2 \sin \dfrac{1}{z}$ $\qquad C: \ |z| = \frac{1}{2}$

g) $f(z) = \dfrac{z^2}{\sin^3 z \cos z}$ $\qquad C: \ |z| = 1$

h) $f(z) = \dfrac{e^z}{z^4+2z^2+1}$ $\qquad C: \ |z-i| = 1$

i) $f(z) = \dfrac{e^{2z}}{z^3-1}$ $\qquad C: \ x^2+y^2 = 2x$

j) $f(z) = \dfrac{\sin(\pi z)}{(z^2-1)^3}$ $\qquad C: \ \frac{x^2}{4} + y^2 = 1$

k) $f(z) = (z+1)e^{\frac{1}{z}}$ $\qquad C: \ |z| = \frac{1}{3}$

l) $f(z) = \operatorname{tg} z$ $\qquad C: \ |z| = 2$

m) $f(z) = \dfrac{e^{\frac{1}{z^2}}}{1+z^2}$ $\qquad C: \ |z-i| = \frac{3}{2}$

10. Izračunaj določeni integral $\int\limits_{-\infty}^{\infty} f(x)\,dx$:

a) $f(x) = \dfrac{1}{(x^2+1)^3}$

b) $f(x) = \dfrac{x}{(x^2+4x+13)^2}$

c) $f(x) = \dfrac{1}{(x^2+2x+2)^2}$

d) $f(x) = \dfrac{1}{(x^2+a^2)(x^2+b^2)}$

e) $f(x) = \dfrac{1}{1+x^6}$

f) $f(x) = \dfrac{\cos 2x}{x^2+9}$

g) $f(x) = \dfrac{x\cos x}{x^2-2x+10}$

h) $f(x) = \dfrac{x\sin x}{1+x^2+x^4}$

11. Izračunaj določeni integral $\int\limits_{0}^{\infty} f(x)\,dx$:

a) $f(x) = \dfrac{1}{(x^2+4)^2}$

b) $f(x) = \dfrac{x^2}{(x^2+4)^2}$

c) $f(x) = \dfrac{x^2+1}{x^4+1}$

d) $f(x) = \dfrac{x\sin x}{x^2+1}$

e) $f(x) = \dfrac{\cos x}{(x^2+1)(x^2+4)}$

f) $f(x) = \dfrac{x^2\cos x}{(x^2+1)^2}$

12. Izračunaj določeni integral $\int\limits_{0}^{2\pi} f(\varphi)\,d\varphi$:

a) $f(\varphi) = \dfrac{1}{(5+4\cos\varphi)^2}$

b) $f(\varphi) = \dfrac{\cos\varphi}{5-\sin\varphi}$

c) $f(\varphi) = \dfrac{1}{1-2a\cos\varphi+a^2}$

$0 < a < 1$

d) $f(\varphi) = \dfrac{1}{2+\cos\varphi}$

e) $f(\varphi) = \dfrac{\cos^2\varphi}{26-10\cos 2\varphi}$

KONFORMNE PRESLIKAVE

1. Poišči linearno preslikavo $w = az + b$, ki preslika

   a) $z_1 = -1 - i \to w_1 = -1 - i$ in $z_2 = 3 - 2i \to w_2 = 3i$

   b) $z_1 = 1 + 2i \to w_1 = 1 + 2i$ in $z_2 = i \to w_2 = -i$

   c) $z_1 = 2 + i \to w_1 = 4 - 3i$ in $z_2 = 1 - i \to w_2 = 1 - i$

2. Poišči linearno preslikavo $w = az + b$, ki preslika trikotnik z oglišči $z_1 = 0$, $z_2 = 1$, $z_3 = i$ na podoben trikotnik z oglišči $w_1 = 0$, $w_2 = 2$, $w_3 = 1 + i$!

3. Poišči preslikavo $w = az + b$, ki preslika območje $\{ z , 3 < \mathrm{Re} z < 5 \}$ na območje $\{ w , 0 < \mathrm{Re} w < 1 \}$!

4. Kaj je slika kroga $x^2 + y^2 - 2x \leq 0$ s preslikavo $w = 3z + i$?

5. Kakšni geometrijski operaciji predstavljata preslikavi $w = iz$ in $w = \frac{1-i}{\sqrt{2}} z$?

6. Poišči preslikavo $w = \frac{az+b}{cz+d}$, ki preslika točke z_1 , z_2 in z_3 v točke w_1 , w_2 in w_3:

   a) $z_1 = -1$ $z_2 = i$ $z_3 = 1 + i$ $w_1 = 0$ $w_2 = 2i$ $w_3 = 1 - i$

   b) $z_1 = -1$ $z_2 = \infty$ $z_3 = i$ $w_1 = \infty$ $w_2 = i$ $w_3 = 1$

   c) $z_1 = -1$ $z_2 = \infty$ $z_3 = i$ $w_1 = 0$ $w_2 = \infty$ $w_3 = 1$

   d) $z_1 = -1$ $z_2 = 0$ $z_3 = 1$ $w_1 = 1$ $w_2 = i$ $w_3 = -1$

7. V katerih točkah ravnine preslikava $w = f(z)$ ni konformna:

   a) $f(z) = z + \frac{1}{z}$

   b) $f(z) = z^3 + 3z$

8. Poišči sliko območja D s preslikavo $w = \frac{1}{z}$:

a) $D = \{ z\,,\ \arg z = \frac{\pi}{3} \}$

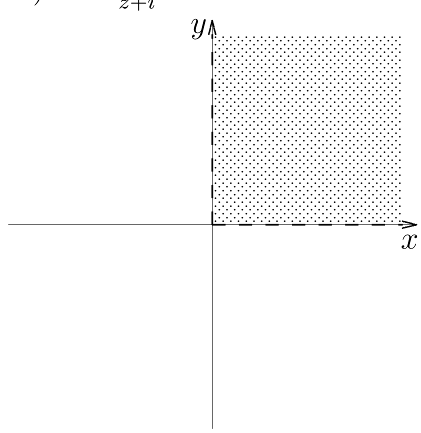
b) $D = \{ z\,,\ |z| = 2 \ \text{ in } \ \frac{\pi}{4} < \arg z < \pi \}$

c) $D = \{ z\,,\ -2 < \text{Im} z < -1 \ \text{ in } \ \text{Re} z = 0 \}$

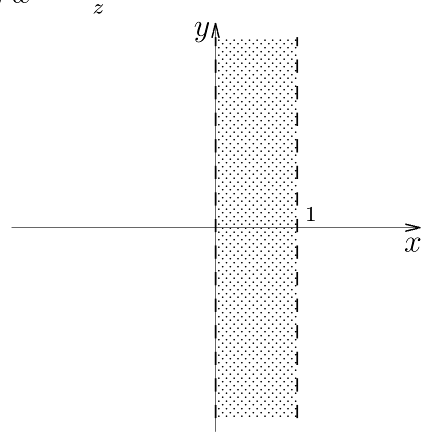
d) $D = \{ z\,,\ |z-1| \leq 1 \}$

9. Poišči sliko danega območja s preslikavo $w = f(z)$:

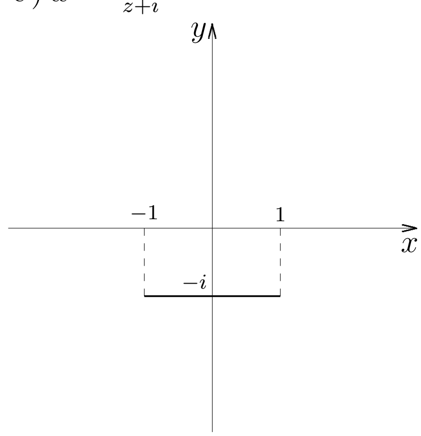
a) $w = \frac{z-i}{z+i}$



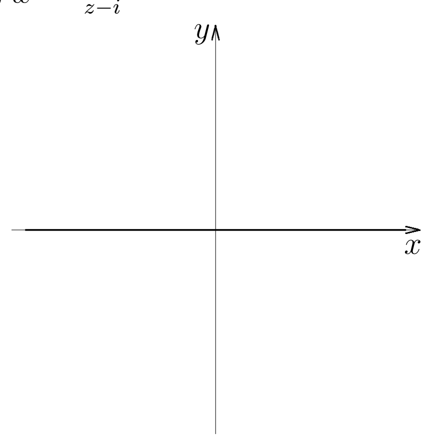
b) $w = \frac{z-1}{z}$



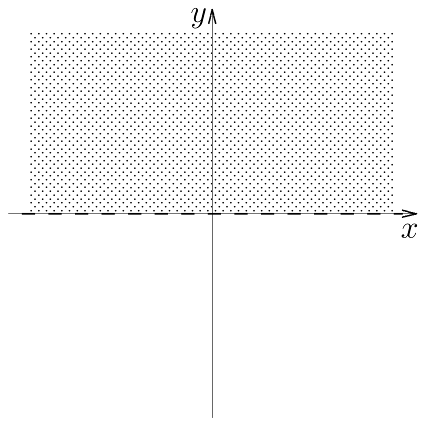
c) $w = \frac{1+iz}{z+i}$



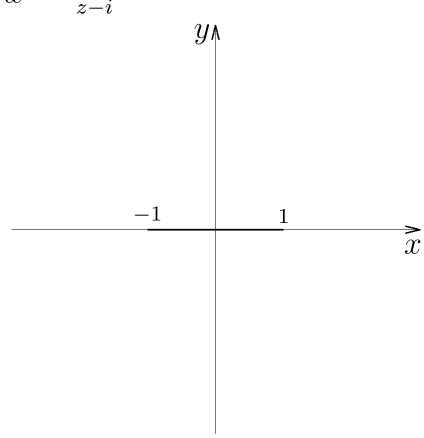
d) $w = \frac{1-iz}{z-i}$



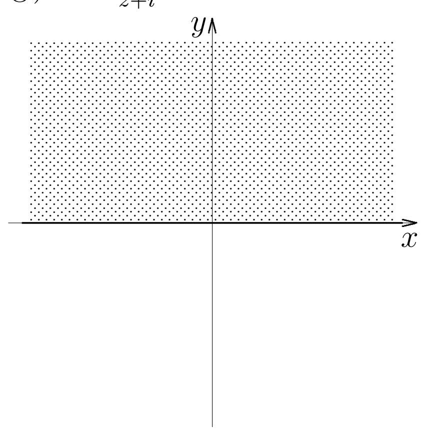
e) $w = (1-i)z$



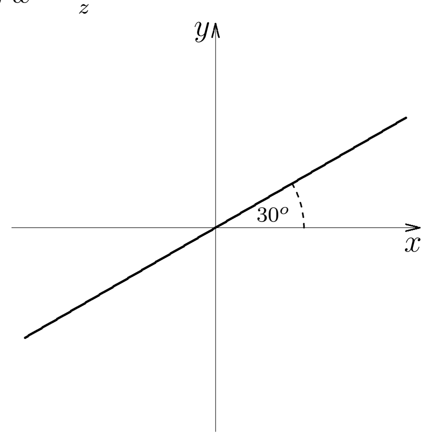
f) $w = \frac{1-iz}{z-i}$



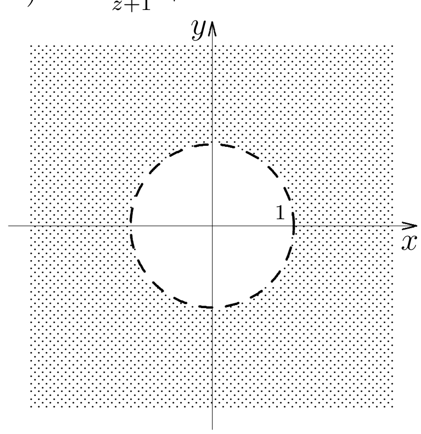
g) $w = \frac{1}{z+i}$



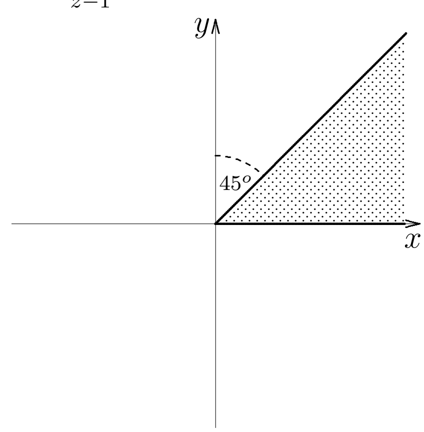
h) $w = \frac{1}{z}$



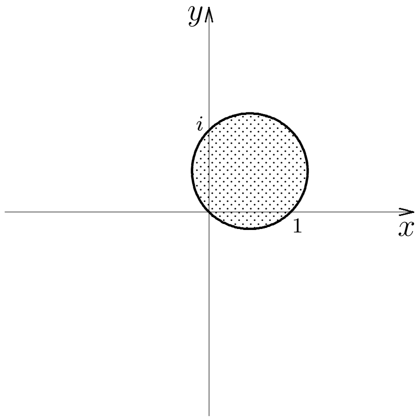
i) $w = \frac{z-1}{z+1} + i$



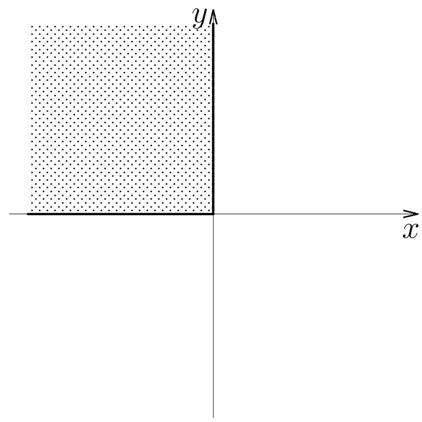
j) $w = \frac{z}{z-1}$



k) $w = \frac{1+i}{2z}$



l) $w = \frac{2iz}{z-1}$



10. Poišči sliko območja

$$\{\, z\,,\ 1 \le |z| \le 2 \ \text{ in } \ \tfrac{\pi}{2} \le \arg z \le \pi \,\}$$

s preslikavo $w = z^2$!

11. Poišči sliko območja

$$\{\, z\,,\ \tfrac{\pi}{6} < \arg z < \tfrac{\pi}{2} \,\}$$

s preslikavo $w = z^3$!

12. Poišči enačbo krivulje v ravnini (w), ki je slika dane daljice oz. premice ravnine (z) s preslikavo $w = e^z$:

a) daljica $x = A\,,\ 0 \le y \le 2\pi$

b) premica $y = B\,,\ -\infty < x < \infty$

Preslikavo prikaži grafično za vrednosti konstant

$A = \ln\frac{3}{4}\,, 0\,, 1$ in $B = 0\,, \frac{2\pi}{3}\,, \frac{3\pi}{2}$!

13. Poišči sliko dane množice v ravnini (z) s preslikavo $w = \sin z$:

a) daljica $\{\, z\,,\ y = C \ \text{ in } \ -\pi \le x \le \pi \,\}$

b) območje $\{\, z\,,\ -\frac{\pi}{2} \le x \le \frac{\pi}{2} \ \text{ in } \ y \ge 0 \,\}$

14. Poišči sliko območja D s preslikavo $w = f(z)$:

a) $w = e^{2z}$ — $D: x < 0$ in $0 < y < \frac{\pi}{4}$

b) $w = e^{\pi i z}$ — $D: 0 < x < 1$

c) $w = 1 + \ln z$ — $D: 1 < |z| < e$ in $0 < \arg z < e$

d) $w = \frac{2i - z^2}{2i + z^2}$ — $D:$ prvi kvadrant

e) $w = (\frac{e^{-z} - 1}{e^{-z} + 1})^2$ — $D: 0 \leq x < \infty$ in $0 \leq y \leq \pi$

f) $w = (\frac{z - i}{z + i})^2$ — $D: |z| \leq 1$ in $x \geq 0$

g) $w = \frac{\sin z}{e^{iz}}$ — $D: \frac{\pi}{6} < x < \frac{\pi}{4}$

h) $w = \ln \frac{z - 1}{z + 1}$ — $D:$ krog skozi točke -1, 1, $\sqrt{3}i$

15. Poišči funkcijo $w = f(z)$, ki preslika kot med poltrakoma

$$z_1 = (1 - i) + te^{\frac{3\pi i}{2}}\,,\ t \geq 0 \quad \text{in} \quad z_2 = (1 - i) + te^{\frac{7\pi i}{4}}\,,\ t \geq 0$$

v prvi kvadrant!

16. Poišči funkcijo $w = f(z)$, ki preslika območje

$$\{\, z\,,\ 0 \leq \arg z \leq \frac{\pi}{4} \,\}$$

v četrti kvadrant!

17. Premici $z_1 = t + ai$ in $z_2 = b + ti$ ($-\infty < t < \infty$, a=konst, b=konst) se sekata pod pravim kotom. Prepričaj se, da se njuni sliki s preslikavo $w = z^2$ tudi sekata pod pravim kotom!

REŠITVE

Diferencialna geometrija

1. a) $\vec{r} = (1-t\,,\,5-3t\,,\,3-4t) \quad , \quad 0 \le t \le 1$

   b) $\vec{r} = (t, t^2, 1) \quad , \quad -\infty < t < \infty$

   c) $\vec{r} = (2\cos t, 2\sin t, 2\sin t + 2\cos t) \quad , \quad 0 \le t \le 2\pi$

   d) $\vec{r} = (t, t, \sqrt{1-2t^2}) \quad , \quad 0 \le t \le \frac{1}{\sqrt{2}}$

   e) $\vec{r}_1 = (\frac{R}{2}\cos t, \frac{R}{2}\sin t, R\frac{\sqrt{3}}{2}) \quad , \quad 0 \le t \le 2\pi$

   $\vec{r}_2 = (t, \pm\sqrt{\frac{R^2}{4} - t^2}, R\frac{\sqrt{3}}{2}) \quad , \quad -\frac{R}{2} \le t \le \frac{R}{2}$

   f) $\vec{r}_1 = (R - \frac{t^2}{R}\,,\, \pm t\sqrt{1-(\frac{t}{R})^2}\,,\, t) \quad , \quad 0 \le t \le R$

   $\vec{r}_2 = (R\sin^2 t\,,\, R\sin t\cos t\,,\, R\cos t) \quad , \quad 0 \le t \le \pi$

   $\vec{r}_3 = (\frac{R}{2}(1+\cos t), \frac{R}{2}\sin t, R\sqrt{\frac{1}{2}(1-\cos t)}) \quad , \quad 0 \le t < 2\pi$

2. a) $99 + \ln 10$

   b) 5

   c) a^2

   d) $\sqrt{9 + 4\pi^2}$

   e) $\ln(2+\sqrt{3})$

   f) $\frac{3}{\sqrt{2}}$

   g) $a\ln(1+\sqrt{2}) = a\ln\operatorname{tg}(\frac{3\pi}{8})$

   h) $\sqrt{3}(e^t - 1)$

   i) 5

   j) 12

   k) $z\sqrt{2}$

   l) $\frac{a}{2}(1 + \ln\sqrt{3})$

   m) 42

3. a) $\vec{r} = (a\cos\frac{s}{\sqrt{a^2+b^2}}\,,\; a\sin\frac{s}{\sqrt{a^2+b^2}}\,,\; \frac{b\,s}{\sqrt{a^2+b^2}})$

b) $\vec{r} = (1+\frac{s}{\sqrt{3}})(\cos\ln(1+\frac{s}{\sqrt{3}})\,,\; \sin\ln(1+\frac{s}{\sqrt{3}})\,,\; 1)$

c) $\vec{r} = (3\sin\frac{s}{5}\,,\; 3\cos\frac{s}{5}\,,\; 2\pi - \frac{4s}{5})$

6. $\vec{r} = (\cos\varphi\,,\; \sin\varphi\,,\; 1-\cos\varphi-\sin\varphi)$

7. $2x+y-z=-1$

8. presek ravnine $x+z=a$ in sfere $(x-\frac{a}{2})^2+y^2+(z-\frac{a}{2})^2=\frac{a^2}{2}$

9. a) $\frac{x-\frac{a}{2}}{a} = \frac{z-\frac{c}{2}}{-c}\,,\; y=\frac{b}{2}$ $\qquad ax-cz=\frac{a^2-c^2}{2}$

b) $x+1-\frac{\pi}{2} = y-1 = \frac{z-2\sqrt{2}}{\sqrt{2}}$ $\qquad x+y+\sqrt{2}z=\frac{\pi}{2}+4$

c) $\frac{x+1}{2} = \frac{y-13}{3} = \frac{z}{6}$ $\qquad 2x+3y+6z=37$

d) $x-2 = \frac{y-4}{4} = \frac{z-8}{12}$ $\qquad x+4y+12z=144$

e) $\frac{x-t^4/4}{t^2} = \frac{y-t^3/3}{t} = z-\frac{t^2}{2}$ $\qquad t^2x+ty+z=\frac{t^6}{4}+\frac{t^4}{3}+\frac{t^2}{2}$

f) $x-1=y-1=\frac{z-1}{2}$ $\qquad x+y+2z=4$

g) $\frac{x-1}{3} = \frac{y-1}{3} = \frac{z-3}{-1}$ $\qquad 3x+3y-z=3$

h) $x=2-z\,,\; y=-2$ $\qquad x-z=0$

i) $\frac{x-1}{12} = \frac{y-3}{-4} = \frac{z-4}{3}$ $\qquad 12x-4y+3z=12$

j) $x-1=y-1=\frac{z+\sqrt{2}}{-\sqrt{2}}$ $\qquad x+y-\sqrt{2}z=4$

k) $x-1=\frac{y-1}{-1}\,,\; z=1$ $\qquad x-y=0$

10. a) $T_1(\frac{1}{4},-\frac{1}{3},\frac{1}{2}) \quad , \quad T_2(4,-\frac{8}{3},2)$

b) $T_1(-1,1,-1) \quad , \quad T_2(-\frac{1}{3},\frac{1}{9},-\frac{1}{27})$

c) $T(\ln 2,4,4)$

11. $\vec{i} + \vec{k}$

12. $T(1, 0, 2)$

13. a) $\vec{r} = (R\cos u\,,\ R\sin u\,,\ v) \quad , \quad 0 \leq u \leq 2\pi\ ,\ 0 \leq v \leq H$

    b) $\vec{r} = (u\cos v\,,\ u\sin v\,,\ u) \quad , \quad 0 \leq v \leq 2\pi\ ,\ 0 \leq u < \infty$

    c) $\vec{r} = (u, u^2, v) \quad , \quad -\infty < u < \infty\ ,\ -\infty < v < \infty$

    d) $\vec{r} = (u, v, u) \quad , \quad -\infty < u < \infty\ ,\ -\infty < v < \infty$

14. a) $z = \frac{x^2+y^2}{2}$

    b) $z = \frac{2xy}{x^2+y^2}$

    c) $x^2 + y^2 + z^2 = a^2 \quad , \quad (z > 0)$

15. a) zgornja polsfera s središčem v izhodišču z radijem a

    b) ploskev ima obliko gladkih spiralnih stopnic

    c) ravnina xy

    d) ravnina xy

    e) plašč valja z višino H in radijem 1

    f) žleb postavljen na pozitivni del osi y

16. $\varphi = \arccos \frac{\sqrt{2}}{3}$

17. $\varphi = \frac{\pi}{2}$

18. a) $\varphi = \frac{1}{3}$

    b) $\varphi_1 = \frac{\pi}{4} \quad , \quad \varphi_2 = \frac{\pi}{2}$

19. a) $ds^2 = (2 + v^2)du^2 + 2uvdudv + (2 + u^2)dv^2$

    b) $ds^2 = du^2 + (1 + u^2)dv^2$

    c) $ds^2 = a^2du^2 + (b - a\cos u)^2dv^2$

    d) $ds^2 = (1 + 4u^2)du^2 + dv^2$

20. $\frac{\pi}{4}$

21. $\frac{\pi}{2}$

22. krožnica $x^2 + y^2 = 2$, $z = 1$

23. a) $x + y - 2z = 0$ $\quad$ $x - 1 = y - 1 = \frac{z-1}{-2}$

b) $2x + 4y - z = 5$ $\quad$ $\frac{x-1}{2} = \frac{y-2}{4} = \frac{z-5}{-1}$

c) $x - y + 2z = \frac{\pi}{2}$ $\quad$ $x - 1 = \frac{y-1}{-1} = \frac{z-\frac{\pi}{4}}{2}$

d) $3x + 4y + 12z = 169$ $\quad$ $\frac{x}{3} = \frac{y}{4} = \frac{z}{12}$

e) $x + y - 4z = 0$ $\quad$ $x - 2 = y - 2 = \frac{z-1}{-4}$

f) $x + 11y + 5z = 18$ $\quad$ $x - 1 = \frac{y-2}{11} = \frac{z+1}{5}$

g) $2x + y + 11z = 25$ $\quad$ $\frac{x-1}{2} = y - 1 = \frac{z-2}{11}$

h) $3x - \sqrt{3}y + 2z = \frac{2\pi}{\sqrt{3}}$ $\quad$ $\frac{x-\frac{1}{2}}{3} = \frac{y-\frac{\sqrt{3}}{2}}{-\sqrt{3}} = \frac{z-\frac{\pi}{\sqrt{3}}}{2}$

i) $-2x + 12y + z = 18$ $\quad$ $\frac{x-5}{-2} = \frac{y-3}{12} = z + 8$

24. premica $\vec{r} = (0, v, 2)$

25. $x - y + 2z = \pm\sqrt{5.5}$

26. $2x + y - z = 2$

27. $A(4, -2, 0)$, $B(-4, 2, 0)$

28. $x + y + z = 3$

29. $T(\frac{a^2}{d}, \frac{b^2}{d}, \frac{c^2}{d})$, $d = \sqrt{a^2 + b^2 + c^2}$

30. $x + 4y + 6z = \pm 21$

31. $\frac{\pi}{4}$

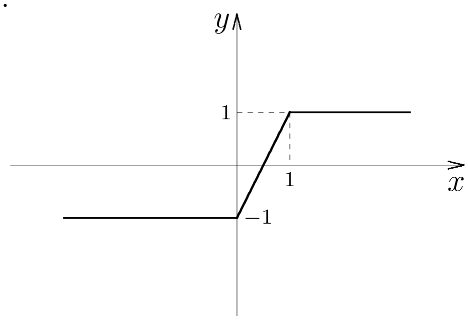
32. $\frac{\pi}{3}$

Integrali s parametrom

1. $(-\infty, 0) \cup (0, \infty)$
2. a) -2

   b) ∞

3.



4. a) $\frac{\cos 2x - \cos x}{x}$

   b) $\frac{2}{x} \ln(1 + x^2)$

   c) $\frac{5 \sin x^5 - 4 \sin x^4}{x}$

   d) $2xe^{-x^5} - e^{-x^3} - \int_x^{x^2} y^2 e^{-xy^2}\, dy$

   e) $(\frac{1}{x} + \frac{1}{b+x}) \sin(bx + x^2) - (\frac{1}{x} + \frac{1}{a+x}) \sin(ax + x^2)$

   f) $-\frac{2}{y} e^{-y^2}$

5. $(n-1)! f(x)$
6. $x(2 - 3y^2) f(xy) + \frac{x}{y^2} f(\frac{x}{y}) + x^2 y(1 - y^2) f'(xy)$
7. $3f(x) + 2xf'(x)$

8. a) $a = -\frac{11}{3}$, $b = 4$

   b) $a = 0.934$, $b = 0.427$

9. $g(x) = \frac{1}{x}$

11. a) minimum pri $y = 2$

   b) maximum pri $x = \ln 2$

   c) maximum pri $x = 0$

14. a) $\operatorname{arcctg} y$

   b) $\pi \arcsin a$

   c) $\frac{\pi}{2} \ln(1 + a)$

   d) $\pi \arcsin a$

   e) $\frac{\pi}{2} \ln(1 + a)$

   f) $\ln(1 + a)$

   g) $\frac{1}{2} \ln(1 + a)$

   h) $\frac{1}{2} \ln \frac{b}{a}$

15. a) $\ln \frac{1+b}{1+a}$

   b) $\frac{\pi}{4\sqrt{a^3}}$

   c) $\frac{3\pi}{16\sqrt{a^5}}$

   d) $\ln \frac{b}{a}$

   e) $\operatorname{arctg} \frac{b}{n} - \operatorname{arctg} \frac{a}{n}$

   f) $\frac{(-1)^n n!}{(1+y)^{n+1}}$

   g) $\frac{\pi+2}{8y^3}$

   h) $\frac{1}{4a}\sqrt{\frac{\pi}{a}}$

Dvojni integral

1. a) $$\int_{-1}^{1} dx \int_{2x^2}^{2} f\,dy \;=\; \int_{0}^{2} dy \int_{-\sqrt{y/2}}^{\sqrt{y/2}} f\,dx$$

b) $$\int_{-3}^{-2} dx \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} f\,dy \;+\; \int_{-2}^{2} dx \int_{-\sqrt{1+x^2}}^{\sqrt{1+x^2}} f\,dy \;+\; \int_{2}^{3} dx \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} f\,dy$$

c) $$\int_{0}^{2} dx \int_{x}^{2x} f\,dy \;+\; \int_{2}^{3} dx \int_{x}^{6-x} f\,dy \;=\; \int_{0}^{3} dy \int_{\frac{y}{2}}^{y} f\,dx \;+\; \int_{3}^{4} dy \int_{\frac{y}{2}}^{6-y} f\,dx$$

d) $$\int_{0}^{1} dy \int_{-y}^{y} f\,dx + \int_{1}^{\sqrt{2}} dy \int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}} f\,dx \;=\; \int_{-1}^{0} dx \int_{-x}^{\sqrt{2-x^2}} f\,dy + \int_{0}^{1} dx \int_{x}^{\sqrt{2-x^2}} f\,dy$$

e) $$\int_{0}^{1} dx \int_{0}^{\sqrt{1-x^2}} f\,dy \;=\; \int_{0}^{1} dy \int_{0}^{\sqrt{1-y^2}} f\,dx$$

f) $$\int_{0}^{1} dx \int_{x-1}^{1-x} f\,dy \;=\; \int_{-1}^{0} dy \int_{0}^{1+y} f\,dx \;+\; \int_{0}^{1} dy \int_{0}^{1-y} f\,dx$$

g) $$\int_{-\sqrt{2}}^{\sqrt{2}} dx \int_{x^2}^{4-x^2} f\,dy \;=\; \int_{0}^{2} dy \int_{-\sqrt{y}}^{\sqrt{y}} f\,dx \;+\; \int_{2}^{4} dy \int_{-\sqrt{4-y}}^{\sqrt{4-y}} f\,dx$$

h) $$\int_{0}^{1} dx \int_{\frac{x}{2}}^{2x} f\,dy \;+\; \int_{1}^{2} dx \int_{\frac{x}{2}}^{\frac{2}{x}} f\,dy \;=\; \int_{0}^{1} dy \int_{\frac{y}{2}}^{2y} f\,dx \;+\; \int_{1}^{2} dy \int_{\frac{y}{2}}^{\frac{2}{y}} f\,dx$$

2. a) $\displaystyle\int_0^1 dy \int_{\arcsin y}^{\pi-\arcsin y} f(x,y)\,dx$

b) $\displaystyle\int_0^{48} dy \int_{\frac{y}{12}}^{\sqrt{\frac{y}{3}}} f(x,y)\,dx$

c) $\displaystyle\int_{\frac{1}{e}}^{1} dy \int_{-\ln y}^{1} f(x,y)\,dx \;+\; \int_1^e dy \int_{\ln y}^{1} f(x,y)\,dx$

d) $\displaystyle\int_{-1}^{0} dx \int_0^{\sqrt{1-x^2}} f(x,y)\,dy \;+\; \int_0^1 dx \int_0^{1-x} f(x,y)\,dy$

e) $\displaystyle\int_{-1}^{0} dy \int_{-\sqrt{y+1}}^{\sqrt{y+1}} f(x,y)\,dx \;+\; \int_0^1 dy \int_{-\frac{2}{\pi}\arccos y}^{\frac{2}{\pi}\arccos y} f(x,y)\,dx$

f) $\displaystyle\int_0^1 dy \int_{-\sqrt{1-y}}^{\sqrt{2y-y^2}-1} f(x,y)\,dx$

g) $\displaystyle\int_0^{R/\sqrt{2}} dy \int_y^{\sqrt{R^2-y^2}} f(x,y)\,dx$

h) $\displaystyle\int_{-6}^{2} dx \int_{\frac{x^2}{4}-1}^{2-x} f(x,y)\,dy$

3. a) $\frac{5}{12}$

b) $-\frac{32}{3}$

c) $\frac{5}{8}(\ln 4 - 1)$

d) $\frac{5}{8}$

e) 0

f) $\frac{2}{3}$

g) $\ln 2$

h) $\frac{4}{3}$

i) $\frac{12}{5}$

j) $\frac{8\sqrt{2}}{21}p^5$

k) $\frac{8}{3}a\sqrt{2a}$

l) $\frac{1}{6}$

m) $\frac{\pi}{6}$

n) $-\frac{1}{2} + \frac{2}{e}$

o) $\frac{4}{3}$

p) $14a^4$

3. q) $\frac{22}{15}$

r) $\frac{5\pi}{2}a^3$

s) $\frac{112}{9}$

t) $\frac{9}{20}$

u) $\frac{\pi^2}{32}$

v) e

w) $\frac{\pi}{32}$

4. a) $\displaystyle\int_0^{2\pi} d\varphi \int_0^1 rf\,dr$

b) $\displaystyle\int_0^{\frac{\pi}{2}} d\varphi \int_0^1 rf\,dr$

c) $\displaystyle\int_0^{\frac{\pi}{2}} d\varphi \int_{1/(\cos\varphi+\sin\varphi)}^{1} rf\,dr$

d) $\displaystyle\int_0^{\frac{\pi}{6}} d\varphi \int_{\sqrt{6}\cos\varphi}^{2\sqrt{\cos 2\varphi}} rf\,dr$

5. a) $\frac{\pi a}{2}$

   b) $\frac{\pi}{2}$

   c) $\frac{2\pi ab}{3}$

   d) $-6\pi^2$

   e) $\frac{1}{4}$

   f) 0

   g) $\frac{1}{4}$

   h) $\frac{4}{3}$

   i) $\frac{\pi}{8}a^4$

   j) 0

   k) $1 - \ln\sqrt{2}$

6. a) $\frac{7}{3}\ln 2$

   b) $\frac{4\pi}{3}\sqrt{\frac{8}{3}}$

   c) $\frac{4}{27}$

7. $\frac{17}{18}$

8. a) $\frac{1}{2}$

   b) $\frac{\pi}{4}$

   c) $-\pi$

   d) $\frac{1}{p-2}$, $(p > 2)$

   e) $\frac{\pi}{p-1}$, $(p > 1)$

   f) π

   g) 2

   h) ∞

   i) $\frac{1}{2}$

9. a) $\frac{10}{3}a^2$

   b) $7\ln 2$

   c) $(\frac{15}{8} - 2\ln 2)a^2$

   d) $\frac{2}{3}$

   e) $\frac{1}{2} - \frac{1}{e}$

   f) πa^2

   g) $3(\frac{\pi}{4} + \frac{1}{2})$

   h) $4a^2$

   i) $\frac{a^2}{6}$

   j) $8\pi + 9\sqrt{3}$

   k) $\pi - 1$

   l) $\pi - 1$

   m) $\frac{7}{16}$

10. a) $\frac{9}{2}$

    b) $(2 + \frac{\pi}{4})a^2$

11. $P = 16a^2$

    $V = \frac{16}{3}a^3$

12. $V = 4\pi\sqrt{3}$

$P = 8\pi\sqrt{3}$

13. a) $\frac{2}{3}R^3$

b) $\pi(1 - e^{-R^2})$

c) $\frac{3}{35}$

d) $\frac{\pi}{8}$

e) $\frac{a^3}{3}$

f) $\frac{48\sqrt{6}}{5}$

g) $\frac{\pi}{3}$

h) $\frac{32a^3}{9}$

i) $\frac{560}{3}$

j) $\frac{45}{32}\pi$

k) $\frac{19}{6}\pi$

14. a) 14

b) 36

c) 8π

d) $\frac{14\pi}{3}$

e) $\frac{\pi}{2}(\sqrt{2} + \ln(1 + \sqrt{2}))$

f) $2R^2(\pi - 2)$

g) $\frac{4}{\sqrt{3}}$

h) $8\sqrt{2}$

15. $T(\frac{4}{5}, 1)$

Trojni integral

1. a) $\frac{a^{11}}{110}$

b) $2e - 5$

c) $\frac{4\pi\sqrt{2}}{3}$

d) $\frac{1}{48}$

2. a) 12

b) $\frac{1}{2}(\ln 2 - \frac{5}{8})$

c) $\frac{1}{180}$

d) $\frac{\pi^2}{16} - \frac{1}{2}$

e) $\frac{(e-1)^3}{6}$

f) $\frac{a^4}{6}$

3. a) $\int_0^2 dx \int_0^{2-x} dy \int_0^{x^2} y\,dz$

b) $\int_0^4 dz \int_{\sqrt{z}}^2 dx \int_0^{2-x} y\,dy$

c) $\frac{8}{15}$

4. a) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} d\varphi \int_0^R r dr \int_0^1 f dz$

b) $\int_0^{\frac{\pi}{2}} d\varphi \int_0^{\frac{\pi}{2}} \cos\vartheta d\vartheta \int_0^R f r^2 dr$

c) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} r dr \int_0^{r^2} f dz$

d) $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\varphi \int_0^{R\sqrt{\cos 2\varphi}} r dr \int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} f dz$

5. $\frac{\pi}{4}(\pi - 2)$

6. a) $\frac{8}{9}a^2$

b) $\frac{4\pi}{15}R^5$

c) $\frac{\pi}{8}$

d) $\frac{\pi(2\sqrt{2}-1)}{15}$

7. a) $\frac{31\pi}{10}$

b) $\frac{8\pi}{3}$

c) $\frac{13\pi}{4}$

d) $\frac{\pi}{10}$

e) $\frac{2\pi}{3}$

f) $\frac{\pi}{10}$

g) $\frac{248\pi}{15}$

h) $\frac{\pi}{6}$

i) 16

j) $16R^3(\frac{\pi}{6} - \frac{2}{9})$

k) $\frac{3}{5}\pi\sqrt{3}$

l) $\frac{59\pi}{480}a^5$

8. a) 8

b) $\frac{\pi}{8}$

c) $\frac{\pi}{3}a^3$

d) $\frac{\pi}{2}$

e) $\frac{22\pi}{3}$

f) $\frac{5\pi}{12}R^3$

g) $\frac{\pi^2}{2}$

h) $\frac{2}{3}\pi a^3(3 - \sqrt{2})$

9. $T(\frac{4}{3\pi}, \frac{4}{3\pi}, \frac{4}{3\pi})$

10. a) $\frac{\pi}{10}$

11. a) $\frac{\pi}{16}$

b) $(\ln R - \frac{1}{3})\frac{8\pi}{3}R^3$

c) $\frac{3}{4}$

d) $\frac{\pi}{6}$

Teorija polja

1. a) stožci $z = C\sqrt{x^2+y^2}$
   b) hiperbolični valji $xy = C$
   c) ravnine $(\vec{a} - C\vec{b}) \cdot \vec{r} = 0$
   d) stožci $z = \sqrt{x^2+y^2}\sin C$
2. $u = 0$ na osi x
   $u = 1$ na ravninah $z = \pm x$
   $u = \sqrt{2}$ na stožcu $z^2 = 2x^2 + y^2$
3. elipsoid $\frac{x^2+y^2}{960} + \frac{z^2}{1024} = 1$
5. $\operatorname{grad} u(A) = (3, -2, -6)$
   $\operatorname{grad} u(B) = 7\vec{i}$
   $\operatorname{grad} u = 0$ v $T(-2, 1, 1)$
6. a) ploskev $z^2 = xy$
   b) premica $x = y = z$
7. $\varphi = \arccos -\frac{8}{9}$
8. $u = xyz - x + y + 2z + C$
9. na sferi $r = 1$
12. $\frac{1}{75}(-2, 4, -4)$
13. ravnina $y + z = 0$
14. a) $-\frac{\sqrt{2}}{9}$
    b) $\frac{1}{3}$
    c) $\frac{98}{13}$
    d) 5
    e) $\frac{\sqrt{2}}{3}$
    f) $\frac{3}{\sqrt{2}}$
15. ravnina $x + y + \sqrt{2}z = 0$
16. 220
17. a) 0
    b) $6(x+z)$
    c) 0
    d) $6xyz$
    e) $\frac{18}{125}$
18. a) $(0, 0, 3y)$
    b) $(-y, 2x, 0)$
    c) $-2(z, x, y)$
    d) $(-\frac{5}{4}, -1, \frac{5}{2})$
    e) $(0, -1, -2)$
    f) $(-16, -9, 0)$
19. a) $\vec{V} = \frac{1}{3}\operatorname{grad}(x^3 + y^3 + z^3 + C)$
    b) $\vec{V} = \operatorname{grad}(xy^2 + x^3 z + C)$
    c) ni pot. polje
    d) ni pot. polje
    e) $\vec{V} = \operatorname{grad}(\frac{x^3y^2}{z} - \frac{x^4}{2} + C)$
21. a) $\frac{2}{r}$
    b) $-\frac{3z}{r}$
    c) $rf'(r) + 3f(r)$
    d) $4\vec{a} \cdot \vec{r}$
    e) $\vec{a} \cdot \vec{b}$
    f) 0
    g) $\frac{1}{r}(-y, x, 0)$
    h) 0
    i) $\frac{f'(r)}{r}\,\vec{r} \times \vec{a}$
    j) $\vec{a} \times \vec{b}$
    k) $\vec{a} \times \vec{r}$
    l) $3r\vec{a} - \frac{\vec{a}\cdot\vec{r}}{r}\,\vec{r}$
    m) 0

21. n) $10r^2$

o) 0

p) $f''(r) + \frac{2}{r} f'(r)$

q) $2\vec{a}$

r) $(f''(r) + \frac{4}{r} f'(r))\,\vec{r}$

22. $f(r) = \frac{r}{2}$

23. $f(r) = \frac{C}{r^3}$

24. $f(r) = \frac{C}{r^2}$

26. $\vec{V} = (yz, 2xz, 0) + \operatorname{grad} u$

u je poljubno skalarno polje

Krivuljni in ploskovni integral

1. a) $\frac{13}{6}$

b) $\frac{\pi}{2} a^3$

c) $4\pi a^2$

d) $(4 - 2\sqrt{2})a^2$

e) $2a^2$

f) $\frac{26}{3}$

g) $\frac{\pi\sqrt{2}}{8} R^5$

h) $\sqrt{2}R^2$

i) $\frac{19}{12}$

j) 88.1

k) $4\sqrt[3]{a^7}$

l) 0

2. 2π

3. 2π

4. $T(\frac{4a}{3\pi}, \frac{4a}{3\pi}, \frac{4a}{3\pi})$

5. a) $I_1 = 4$

$I_2 = 4$

b) $\frac{ab}{2}$

c) $-2\sin 2$

d) $\frac{4}{5}$

e) 0

f) -2π

g) $\frac{\pi}{4} - 1$

h) $\frac{112}{3}$

i) $\frac{1}{35}$

j) $-\pi a^2$

k) -2

l) 0

m) $(\frac{1}{6} + \frac{\pi\sqrt{2}}{16})R^3$

n) 6

o) 0

p) $-\frac{5}{6}$

6. a) $-e-1$
   b) 4
   c) $\ln\frac{13}{5}$
   d) $\frac{3}{2}$
   e) -2
   f) 2
   g) 0
7. a) $U=\frac{x^3y^2}{z}-\frac{1}{2}x^4+\frac{3}{4}y^4+\frac{1}{4}z^4+C$
   b) $U=-\frac{1}{r^2}+C$
   c) $U=\operatorname{arctg}(xyz)+C$
   d) $U=\frac{x-3y}{z}+\frac{1}{2}z^2+C$
   e) $U=x-\frac{x}{y}+\frac{xy}{z}+C$
   f) $U=(x^2-y^2)^2+C$
   g) $U=\ln|x-y|+\frac{y}{x-y}+\frac{x^2}{2}-\frac{y^3}{3}+C$
9. a) $2\pi RH(R^2+\frac{1}{3}H^2)$
   b) $(1+\sqrt{2})\frac{\pi}{2}$
   c) $\sqrt{3}(\ln 2-\frac{1}{2})$
   d) π
   e) $\frac{8}{3}\pi a^4$
   f) $\frac{2}{3}\pi a^2\sqrt{a^2+b^2}$
   g) $2\sqrt{3}a^4$
   h) 0
   i) $\frac{125\sqrt{5}-1}{420}$
   j) $\frac{64\sqrt{2}}{15}a^4$
   k) $\frac{9\sqrt{3}-8\sqrt{2}+1}{15}$
   l) $(3\sqrt{3}-1)\pi$
   m) $\frac{1023}{5}\pi\sqrt{2}$
10. a) 2
   b) $8\pi\sqrt{6}$
   c) $\frac{\pi}{12}$
11. $(1+6\sqrt{3})\frac{2\pi}{15}$
12. $\frac{4\pi\rho}{3}a^4$
13. 8
14. a) 8
   b) $\frac{4\pi}{3}abc$
   c) $\frac{\pi}{2}a^4$
   d) 0
   e) $4\pi a^3$
   f) $R^2H(\frac{2}{3}R+\frac{\pi}{8}H)$
   g) $\frac{\pi}{8}$
   h) $\frac{567}{2}\pi$
15. a) 0
   b) $-4\pi a^2$
   c) $-\frac{9}{2}a^3$
   d) $-\pi a^2$
   e) -1
16. a) $\frac{1}{2}a^3$
   b) $\frac{1}{4}a^4$
17. $\frac{8\pi}{3}(a+b+c)R^3$
18. $-\pi$
20. a) $\frac{12\pi}{5}a^5$
   b) $\frac{\pi}{2}H^4$
   c) $3a^4$
   d) 4π
   e) $\frac{4\pi}{15}$

21. a) 4π
    b) $\frac{1}{2}$
    c) 0
    d) 4
    e) $2\pi\sqrt{3}$

22. a) 4π
    b) 2π

23. 16π

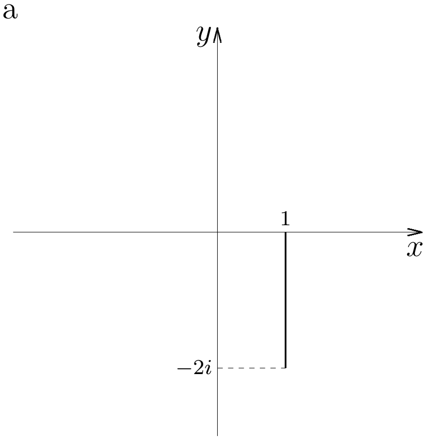
24. 0

26. $\frac{\pi}{2}R^4$

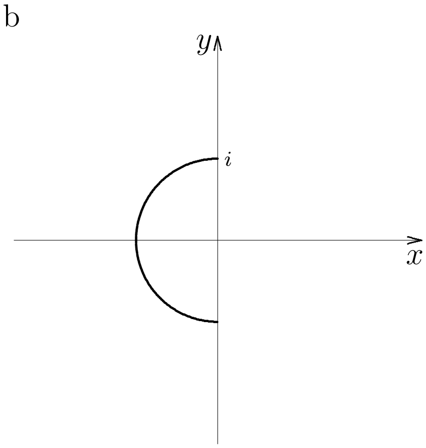
28. $\frac{1}{3}$

Analitične funkcije

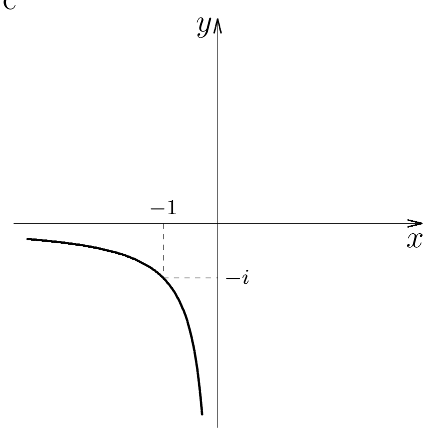
1. a



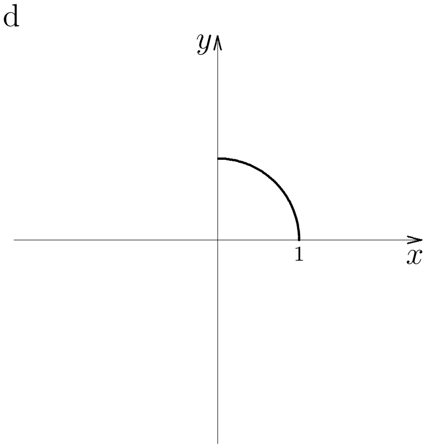
1. b



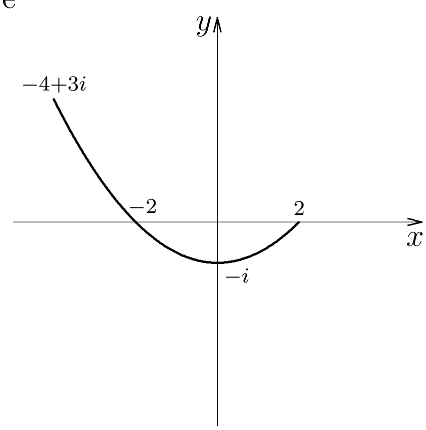
1. c



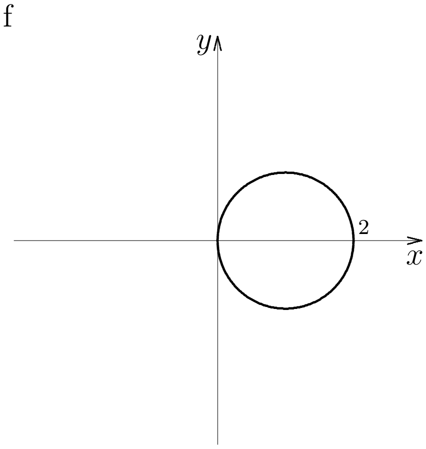
1. d



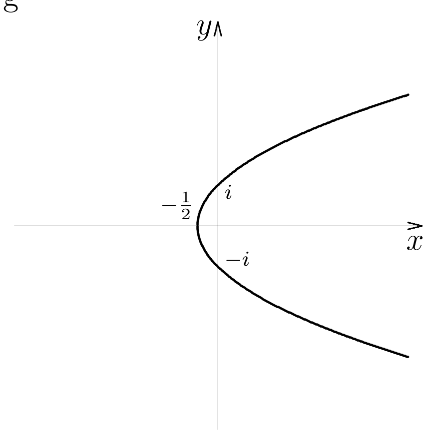
1. e



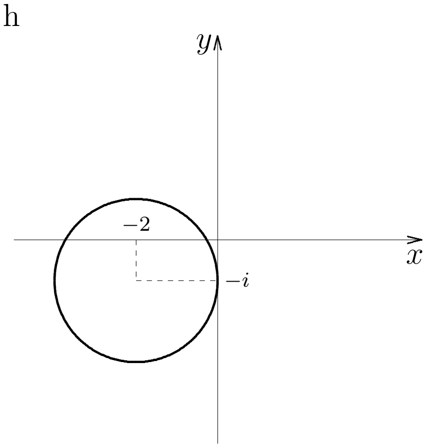
1. f



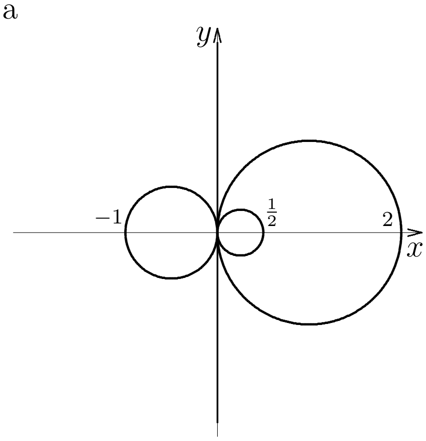
1. g



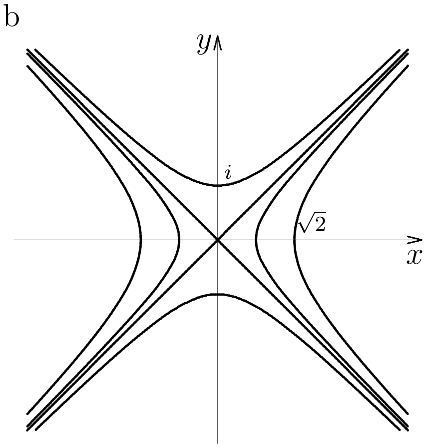
1. h



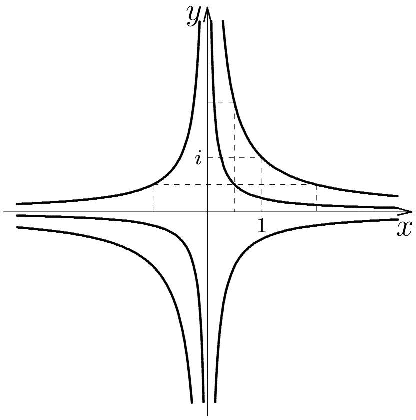
2. a



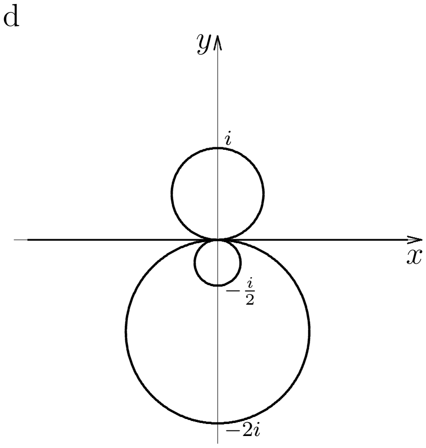
2. b



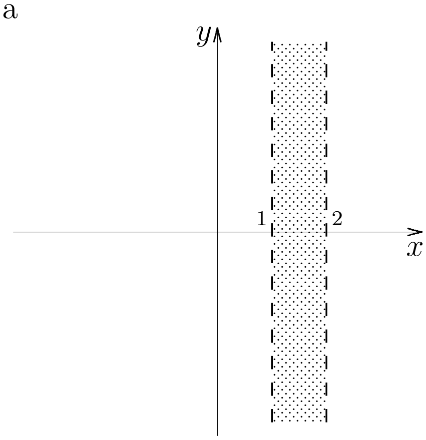
2. c



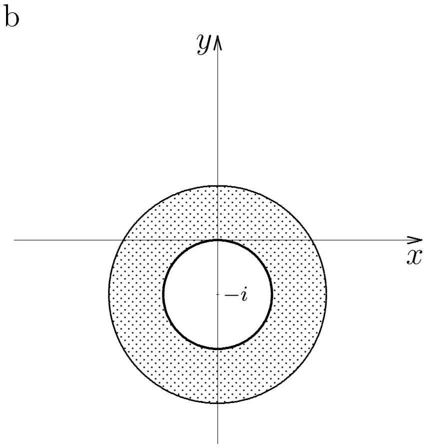
2. d



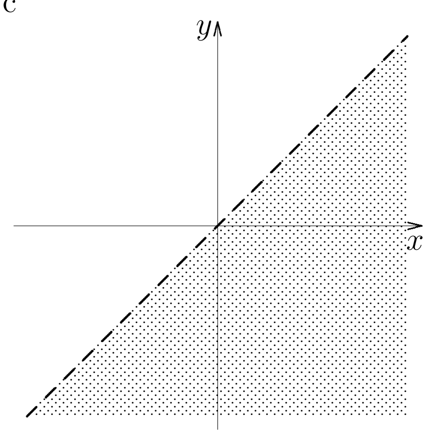
3. a



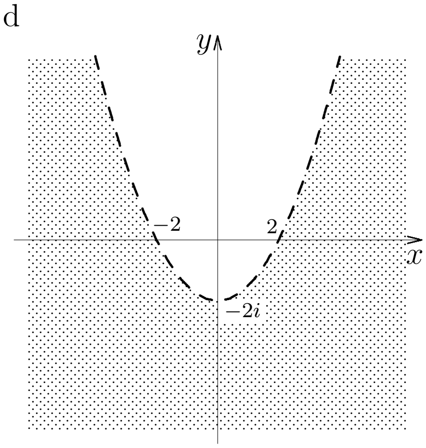
3. b



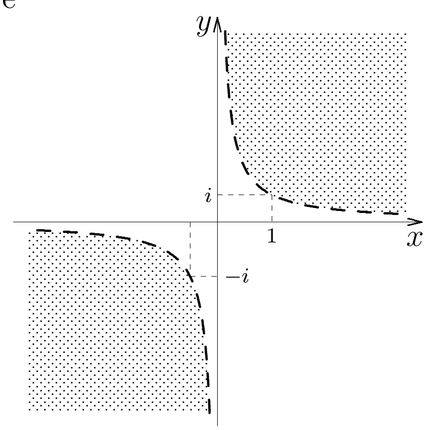
3. c



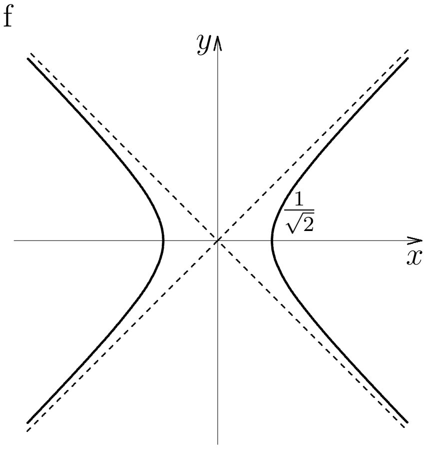
3. d



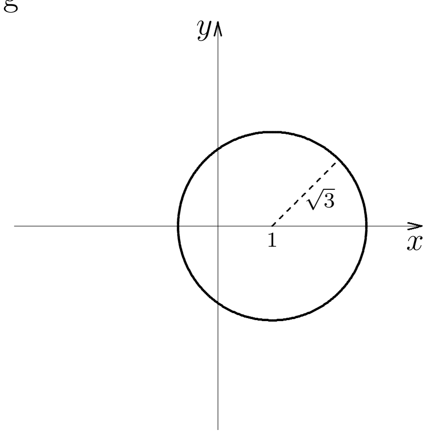
3. e



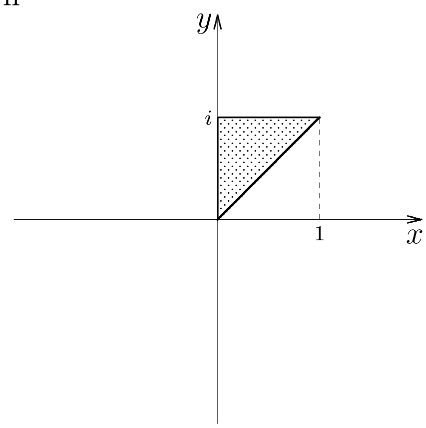
3. f



3. g



3. h



4. a) $u = x^3 - 3xy^2 - y$

$v = 3x^2y - x - y^3$

b) $u = \frac{x^2-y^2}{x^2+y^2}$

$v = \frac{-2xy}{x^2+y^2}$

c) $u = x$

$v = -y$

5. a) $u = 1 - \frac{v^2}{4}$

b) $u = v^2 - \frac{1}{4}$

c) $u = 0\ ,\ v \geq 0$

d) $u^2 + v^2 = \frac{u}{C}$

e) $u^2 + v^2 + \frac{v}{C} = 0$

f) $u = \frac{1}{C}$

g) $v = -Cu$

h) $v = u$

i) $u = -v^2$

6. a) da
   b) ne
   c) ne
   d) da

7. a) da
   b) da
   c) ne
   d) ne

8. a) ne
   b) da
   c) ne
   d) da

9. a) ne
   b) da
   c) da
   d) ne

10. a) $v = 2xy + y + C$
    b) $v = \frac{-y}{x^2+y^2} + C$
    c) $u = \ln\sqrt{x^2 + y^2} + C$
    d) $v = \frac{y^2-x^2}{2}$

11. a) $z^2 + (5-i)z - \frac{i}{z} + Ci$
    b) $\frac{1}{2z} + iz^2 + 3i + C$
    c) $2i\ln z - (2-i)z + C$
    d) $2iz - z^2 + C$
    e) e^{z^2}
    f) $ze^{-z} + i$
    g) $\frac{i-1}{z}$
    h) $3iz + z^2$

12. a) $c = 1\ ,\ b = -a$

13. a) ne
    b) $f(u) = au + b$

14. a) $f(t) = a\,\mathrm{arctg}\,t + b$
    b) $f(t) = a\ln t + b$

15. $u = a(x^2 - y^2) + bx + cy + d$

Elementarne funkcije

1. a) $\sqrt{\frac{e}{2}}(1-i)$
   b) $\frac{8+15i}{17}$
   c) $\frac{40+9i}{41}$
   d) $\frac{8n+1}{4}\pi i$
   e) $\ln\sqrt{13}+i(2n\pi-\operatorname{arctg}\frac{3}{2})$
   f) $e^{2n\pi}$
   g) $(\frac{1-i}{\sqrt{2}})e^{(2n+\frac{1}{4})\pi}$
   h) $2n\pi+i\ln(\sqrt{2}+1)$
   $(2n+1)\pi+i\ln(\sqrt{2}-1)$
   i) -7
   j) $-3.17+1.96i$
   k) $e^{-(2n+\frac{1}{2})\pi}$
   l) $e^{(2n+\frac{3}{2})\pi}$
   m) 1
   n) $-\frac{3}{4}i$
   o) 2
   p) $e^{\sqrt{2}(2n+1)\pi i}$
   q) $e^{-2n\pi}$
   r) $\ln\sqrt{2}+\frac{8n-3}{4}\pi i$
   s) $i\operatorname{th}\frac{\pi}{2}$
   t) $\cos e+i\sin e$

3. a) $\sin z=\sin x\operatorname{ch}y+i\cos x\operatorname{sh}y$
   b) $\cos z=\cos x\operatorname{ch}y-i\sin x\operatorname{sh}y$
   c) $\operatorname{ch}(z-i)=\operatorname{ch}x\cos(y-1)+$
   $i\operatorname{sh}x\sin(y-1)$

4. a) $\operatorname{Im} f(x+n\pi i)=0$
   $\operatorname{Re} f(x+(n+\frac{1}{2})\pi i)=0$
   b) $\operatorname{Im} f((n+\frac{1}{2})\pi+iy)=0$
   $\operatorname{Im} f(x)=0$
   $\operatorname{Re} f(n\pi+iy)=0$
   c) $\operatorname{Im} f(x+n\pi i)=0$
   $\operatorname{Im} f(iy)=0$
   $\operatorname{Re} f(x+(n+\frac{1}{2})\pi i)=0$
   d) $\operatorname{Im} f(n\pi+iy)=0$
   $\operatorname{Im} f(x)=0$
   $\operatorname{Re} f((n+\frac{1}{2})\pi+iy)=0$

6. a) $\frac{2n+1}{4}\pi+iy$
   b) $\frac{8n+1}{4}\pi+i\ln(\sqrt{2}\pm1)$
   c) $\frac{8n+1}{4}\pi+i\ln\frac{\sqrt{6}+\sqrt{2}}{2}$
   $\frac{8n-3}{4}\pi-i\ln\frac{\sqrt{6}+\sqrt{2}}{2}$
   d) $-\ln 2+\frac{4n+1}{2}\pi i$
   e) $\frac{4n-1}{2}\pi$
   f) $2n\pi i$
   $\ln 3+i(2n+1)\pi$
   g) $2n\pi\pm i\ln(2+\sqrt{3})$
   h) $(2n+1)\pi\pm i\ln 2$
   i) $-e+i$
   j) $\frac{4n+1}{2}\pi+i\ln(3\pm\sqrt{8})$
   k) $\frac{4n-1}{2}\pi i$
   l) $x+i\ln 2$
   m) $\pm\sqrt{n\pi}(1\pm i)$

Integral

1. a) $I_1 = -2 + 2i$

   $I_2 = -2 + \frac{4}{3}i$

   $I_3 = -2$

   b) $-\frac{8}{3}$

   c) $(e^\pi + 1)i$

   d) $-\frac{4}{3}$

   e) $I_1 = \sqrt{5}(1 - \frac{i}{2})$

   $I_2 = 8i$

2. a) $7 + 19i$

   b) $\frac{1}{e} - 1$

   c) $-1.38 - 0.3i$

   d) $0.46 - 0.16i$

   e) $2.36 + 1.44i$

   f) $-1 - i \operatorname{sh} 1$

   g) $-2 - 2i$

   h) 0

   i) $0.64 + 0.82i$

   j) $-3.06 + 0.76i$

   k) $2 - 2i$

   l) $\frac{\pi^4}{64}$

   m) $-\frac{\pi^2}{8}$

3. a) 0

   b) $-\frac{\pi}{3}i$

   c) πi

   d) $\frac{\pi}{e}$

   e) $\frac{\pi}{2}i$

   f) $\pi \operatorname{sh} 1$

   g) $\frac{2\pi e^2}{3}i$

Laurentova vrsta

1. a) $|z| < \frac{1}{\sqrt{2}}$
   b) $|z| < 1$
   c) $|z| < \sqrt{2}$
   d) C
   e) $|z| < 2$
   f) $|z| < \frac{1}{e}$
   g) $|z| > \sqrt{2}$
   h) $|z + i| > e$
   i) $|z| > e$
   j) $|z - 2 - i| > \frac{1}{2}$
   k) $|z + 2i| > 3$

1. l) $|z + 1 + i| < 1$
   m) $|z| < 1$
   n) $z = 0$

2. a) $5 < |z + 2i| < 6$
   b) $0 < |z - 2 + i| < 1$
   c) $2 < |z| < 4$
   d) $\emptyset$
   e) $1 < |z| < 2$
   f) $|z + 1| > 2$
   g) $0 < |z| < 1$

3. a) $-\frac{1}{2} - \frac{z}{4} - \frac{z^2}{8} - \frac{z^3}{16} - \frac{z^4}{32} - \cdots$ $\quad R = 2$
   b) $-\frac{1}{3}z + \frac{2}{9}z^2 - \frac{7}{27}z^3 + \cdots$ $\quad R = 1$
   c) $-\frac{1}{5} - \frac{9}{25}z - \frac{41}{125}z^2 - \cdots$ $\quad R = 1$
   d) $-iz + z^3 + iz^5 - z^7 - iz^9 + \cdots$ $\quad R = 1$
   e) $1 + \frac{1}{2}(\frac{z^2}{2!} + \frac{z^4}{4!} + \frac{z^6}{6!} + \cdots)$ $\quad R = \infty$
   f) $\ln 2 - \frac{1}{2}(\frac{z}{2} + \frac{z^2}{8} + \frac{z^3}{24} + \cdots)$ $\quad R = 2$
   g) $z + z^2 + \frac{z^3}{3} - \frac{z^5}{30} - \frac{z^6}{90} + \cdots$ $\quad R = \infty$
   h) $z + \frac{z^3}{3} + \cdots$ $\quad R = \frac{\pi}{2}$

4. a) $\sum\limits_{n=0}^{\infty} \frac{(z-3i)^n}{(1-3i)^{n+1}}$ $\quad |z - 3i| < \sqrt{10}$
   b) $-\sin 1 + 2\cos 1(z+1) + \frac{2^2 \sin 1}{2!}(z+1)^2 - \frac{2^3 \cos 1}{3!}(z+1)^3 - \cdots$ $\quad R = \infty$
   c) $-\sum\limits_{n=0}^{\infty} \frac{(z-\pi i)^n}{n!}$ $\quad R = \infty$
   d) $(z-1) + \sum\limits_{n=2}^{\infty} \frac{(-1)^n}{n(n-1)}(z-1)^n$ $\quad |z - 1| < 1$

5. a) $z_1 = 0$ $\quad -\sum_{n=-1}^{\infty} z^n$ $\quad 0 < |z| < 1$

$z_2 = 1$ $\quad \sum_{n=-1}^{\infty} (-1)^{n+1}(z-1)^n$ $\quad 0 < |z-1| < 1$

b) $z_1 = 0$ $\quad \sum_{n=-1}^{\infty} (-1)^{n+1} z^n$ $\quad 0 < |z| < 1$

$z_2 = -1$ $\quad -\sum_{n=-1}^{\infty} (z+1)^n$ $\quad 0 < |z+1| < 1$

c) $z_1 = i$ $\quad \frac{1}{4}\sum_{n=-1}^{\infty} (\frac{1}{2})^n (z-i)^n$ $\quad 0 < |z-i| < 2$

$z_2 = -i$ $\quad \frac{1}{4}\sum_{n=-1}^{\infty} (\frac{1}{2i})^n (z+i)^n$ $\quad 0 < |z+i| < 2$

d) $z_1 = 1$ $\quad \frac{1}{z-1} - \sum_{n=0}^{\infty} (z-1)^n$ $\quad 0 < |z-1| < 1$

$z_2 = 2$ $\quad \frac{1}{z-2} + \sum_{n=0}^{\infty} (-1)^n (z-2)^n$ $\quad 0 < |z-2| < 1$

6. a) $\sum_{n=2}^{\infty} \frac{(-1)^n}{z^n}$

b) $\sum_{n=2}^{\infty} \frac{(-1)^n}{(z-1)^n}$

c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{z^n} + \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} z^n$

d) $-\frac{1}{3}\sum_{n=1}^{\infty} \frac{1}{z^{2n-1}} - \frac{1}{12}\sum_{n=0}^{\infty} \frac{z^{2n+1}}{4^n}$

e) $\sum_{n=1}^{\infty} \frac{1}{(z+2)^n} - \sum_{n=0}^{\infty} \frac{(z+2)^n}{3^{n+1}}$

7. a) $\sum_{n=0}^{\infty}(\frac{1}{2^{n+1}}-\frac{1}{3^{n+1}})z^n$

b) $-\sum_{n=1}^{\infty}\frac{2^{n-1}}{z^n}-\sum_{n=0}^{\infty}\frac{z^n}{3^{n+1}}$

c) $\sum_{n=2}^{\infty}\frac{3^{n-1}-2^{n-1}}{z^n}$

d) $-\sum_{n=0}^{\infty}(1+(-\frac{1}{2})^{n+1})z^n$

e) $\sum_{n=1}^{\infty}\frac{1}{z^n}+\sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}z^n$

f) $\sum_{n=1}^{\infty}(1+(-2)^{n-1})z^{-n}$

8. $\frac{1}{4}\frac{1}{(z-1)^2}-\frac{1}{4}\frac{1}{z-1}+\frac{3}{16}-\frac{1}{8}(z-1)+\frac{5}{64}(z-1)^2-\cdots$

9. a) $\sum_{n=0}^{\infty}(-1)^{n+1}\frac{(z-\pi)^{2n}}{(2n+1)!}$

b) $\frac{1}{z+1}-\frac{1}{(z+1)^2}$

c) $(z+i)+(1-i)+\sum_{n=1}^{\infty}(\frac{1}{(n+1)!}-\frac{i}{n!})\frac{1}{(z+i)^n}$

d) $(z-i)+\frac{1-i}{(z-i)}-\sum_{n=2}^{\infty}\frac{i}{(n-1)!}\frac{1}{(z-i)^n}$

10. a) $\frac{1}{z}-\frac{z}{3!}+\frac{z^3}{5!}-\frac{z^5}{7!}+\frac{z^7}{9!}-\cdots$

b) $z-\frac{2^3}{4!}z^3+\frac{2^5}{6!}z^5-\frac{2^7}{8!}z^7+\cdots$

c) $\frac{1}{z^3}+\frac{1}{z^2}+\frac{1}{2z}+\frac{1}{3!}+\frac{z}{4!}+\frac{z^2}{5!}+\cdots$

d) $z^3+z^2+\frac{z}{2!}+\frac{1}{3!}+\frac{1}{4!z}+\frac{1}{5!z^2}+\cdots$

e) $\frac{2}{z^4}-\frac{1}{2!z^2}+\frac{1}{4!}-\frac{z^2}{6!}+\frac{z^4}{8!}-\cdots$

f) $\frac{1}{z^2}-\frac{1}{2!z}+\frac{1}{3!}-\frac{z}{4!}+\frac{z^2}{5!}-\cdots$

g) $\frac{1}{z^2}+\frac{1}{z}+1+z+z^2+z^3+\cdots$

h) $z^2-\frac{1}{2}+\frac{1}{4!z^2}-\frac{1}{6!z^4}+\frac{1}{8!z^6}-\cdots$

Singularne točke in uporaba

1. a) 0 2.stopnje
   $\pm 2i$ 1. ”
   b) 0 2. ”
   $\pm\sqrt{n\pi}$ 1. ”
   c) 0 3. ”
   $n\pi$ 1. ”
   d) $(2n+1)\pi$ 6. ”
   e) $2n\pi i$ 1. ”
   f) $\pm\pi i$ 2. ”
   $(2n+1)\pi i$ 1. ”
   g) 1 3. ”
   -1 1. ”

2. a) 4. stopnje
   b) 2. ”
   c) 15. ”
   d) 1. ”

3. a) -1 pol 2. stopnje
   1 pol 1. stopnje
   b) 0 pol 3. stopnje
   c) 0 bistv. sing. točka
   d) $\frac{1}{2}$ pol 1. stopnje
   $\frac{4n+1}{2}$ poli 2. stopnje
   e) -2 bistv. sing. točka
   f) 0 pol 2. stopnje
   $1 \pm i$ pola 1. stopnje
   g) i bistv. sing. točka
   $-i$ pol 1. stopnje
   h) 0 pol 4. stopnje
   -1 pol 1. stopnje
   i) 0 odpravljiva sing.
   $\frac{2n+1}{2}\pi$ poli 1. stopnje

4. a) odpravljiva singularnost
   b) pol 3. stopnje
   c) pol 1. ”
   d) pol 1. ”
   e) pol 4. ”
   f) odpravljiva singularnost
   g) pol 2. stopnje
   h) pol 2. ”
   i) bistv. sing točka

5. a) pol 2. stopnje
   b) pol 1. ”

6. a) regularna točka
   b) regularna točka
   c) ničla 2. stopnje
   d) pol 2. stopnje
   e) bistv. sing točka
   f) pol 2. stopnje

7. a) $\operatorname{res}(-1) = \frac{2}{27}$
   $\operatorname{res}(2) = -\frac{1}{27}$
   b) $\operatorname{res}(2) = 5$
   c) $\operatorname{res}(i) = -\frac{i}{4}$
   $\operatorname{res}(-i) = \frac{i}{4}$
   d) $\operatorname{res}(i) = -1$
   e) $\operatorname{res}(3i) = -\frac{i}{6e^3}$
   $\operatorname{res}(-3i) = \frac{ie^3}{6}$
   f) $\operatorname{res}\left(\frac{2n+1}{2}\pi\right) = -1$
   g) $\operatorname{res}(0) = -\frac{4}{\pi^2}$
   $\operatorname{res}\left(\frac{\pi}{2}\right) = 0$
   h) $\operatorname{res}(i) = \frac{(-1+3i)\cos 1}{20}$
   $\operatorname{res}(-i) = \frac{(-1-3i)\cos 1}{20}$
   $\operatorname{res}(3) = \frac{\operatorname{ch} 3}{10}$
   i) $\operatorname{res}(0) = 0$
   j) $\operatorname{res}(0) = \frac{5}{6}$
   k) $\operatorname{res}(0) = e - 1$
   $\operatorname{res}(1) = -e$

8. 2

9. a) 0
   b) 2π
   c) 0
   d) $\frac{2(e-1)\pi}{e}i$
   e) $-\frac{4\pi \ln 3}{3}i$
   f) $-\frac{\pi}{3}i$
   g) $2\pi i$
   h) $2.17 + 0.47i$
   i) $\frac{2\pi e^2}{3}i$
   j) 0
   k) $3\pi i$
   l) $-4\pi i$
   m) $\frac{\pi}{e}$

10. a) $\frac{3\pi}{8}$
    b) $-\frac{\pi}{27}$
    c) $\frac{\pi}{2}$
    d) $\frac{\pi}{ab(a+b)}$
    e) $\frac{2\pi}{3}$
    f) $\frac{\pi}{3e^6}$
    g) $\frac{\pi(\cos 1 - 3\sin 1)}{3e^3}$
    h) $\frac{\pi e^{-\sqrt{3}/2}}{\sqrt{3}} \sin\frac{1}{2}$

11. a) $\frac{\pi}{32}$
    b) $\frac{\pi}{8}$
    c) $\frac{\pi}{\sqrt{2}}$
    d) $\frac{\pi}{2e}$
    e) $\frac{\pi(2e-1)}{12e^2}$
    f) 0

12. a) $\frac{10\pi}{27}$
    b) 0
    c) $\frac{2\pi}{1-a^2}$
    d) $\frac{2\pi}{\sqrt{3}}$
    e) $\frac{\pi}{20}$

Konformne preslikave

1. a) $w = iz - 2$

   b) $w = (2+i)z + (1-3i)$

   c) $w = \frac{-1-8i}{5}z + \frac{14+2i}{5}$

2. $w = (1+i)(1-z)$

3. $w = \frac{z-3}{2}$

4. $(u-3)^2 + (v-1)^2 \leq 9$

5. w_1 je vrtenje za 90^o okrog izhodišča

   w_2 je vrtenje za 45^o okrog izhodišča

6. a) $w = \frac{-2iz-2i}{4z-1-5i}$

   b) $w = \frac{iz+2+i}{z+1}$

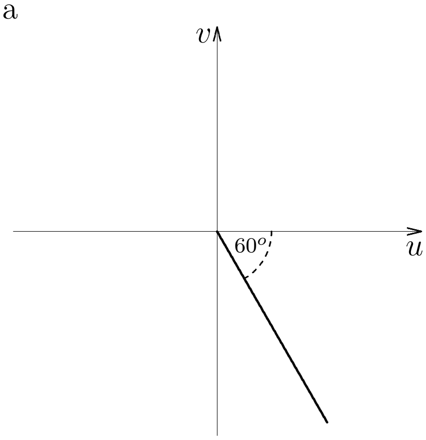
   c) $w = \frac{(1-i)(z+1)}{2}$

   d) $w = \frac{z-i}{iz-1}$

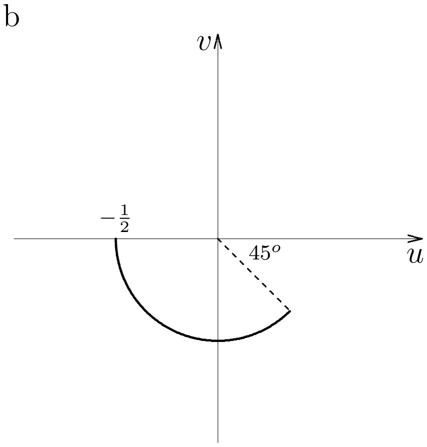
7. a) ± 1

   b) $\pm i$

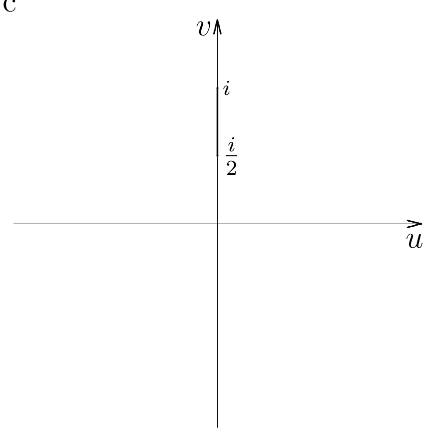
8. a



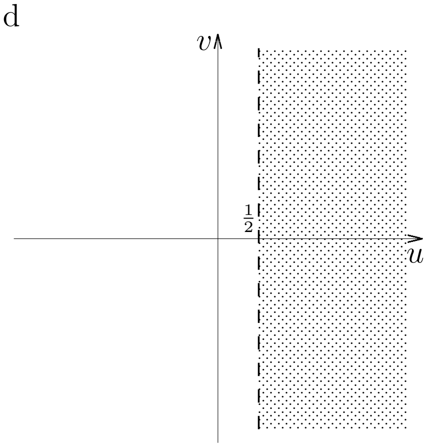
8. b



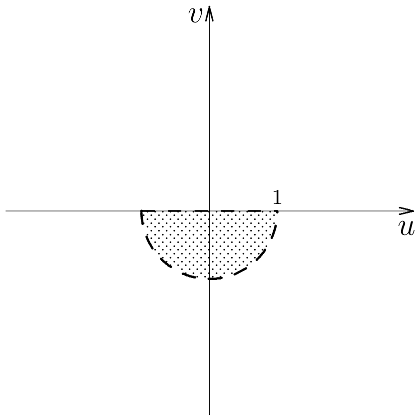
8. c



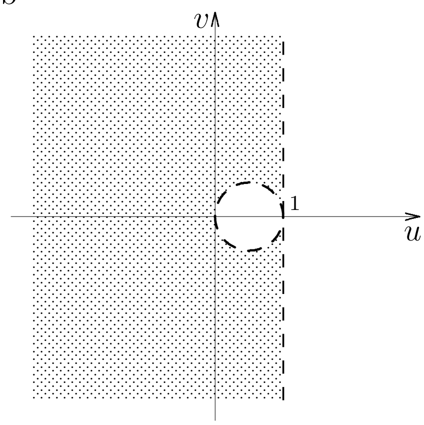
8. d



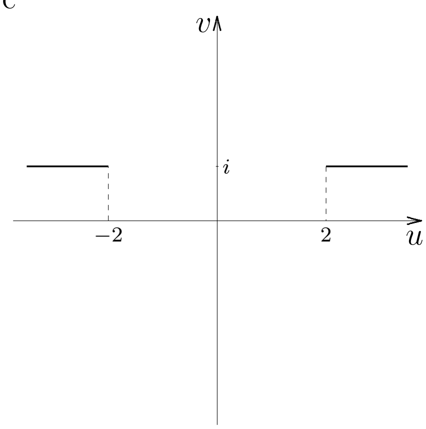
9. a



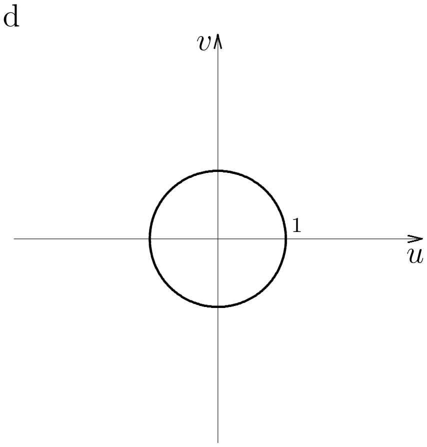
9. b



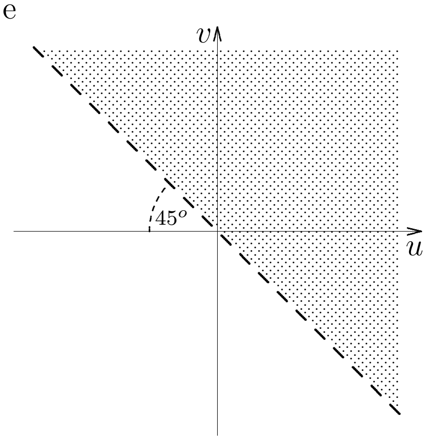
9. c



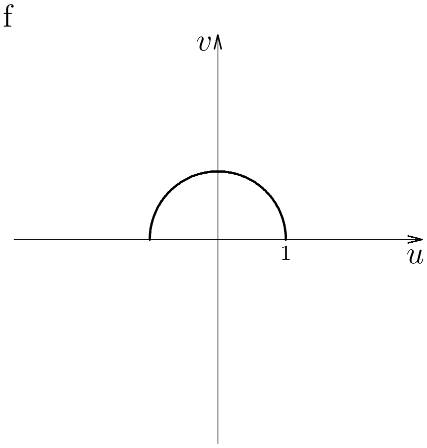
9. d



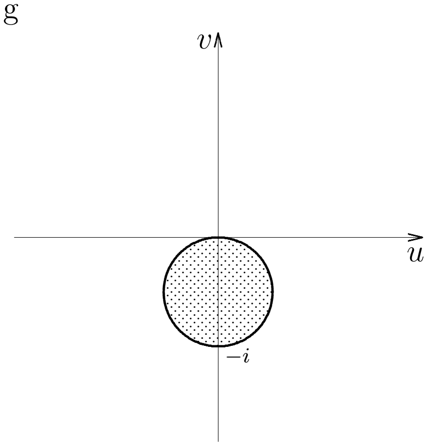
9. e



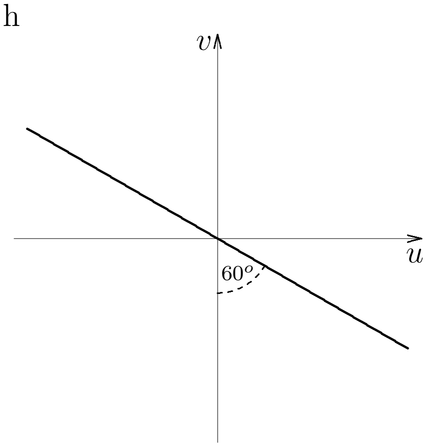
9. f



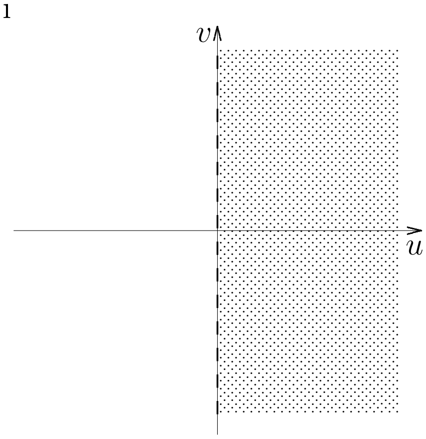
9. g



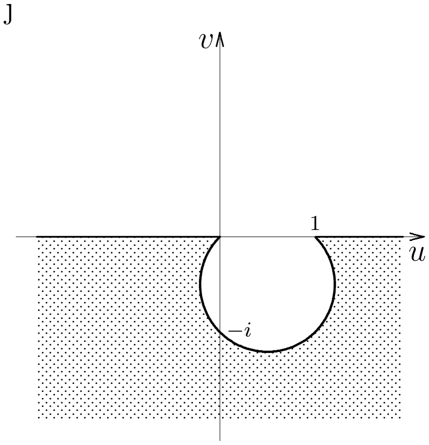
9. h



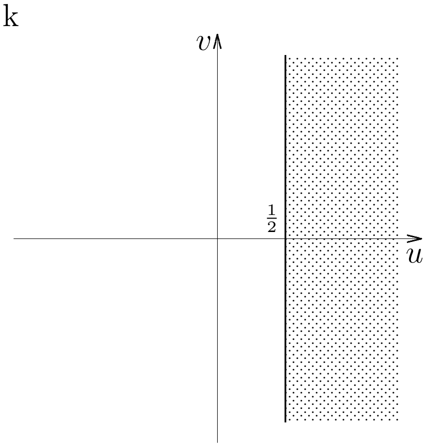
9. i



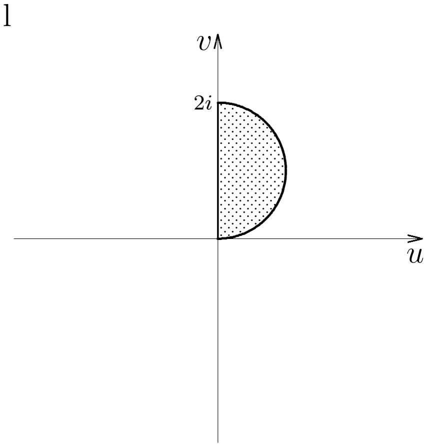
9. j



9. k



9. l



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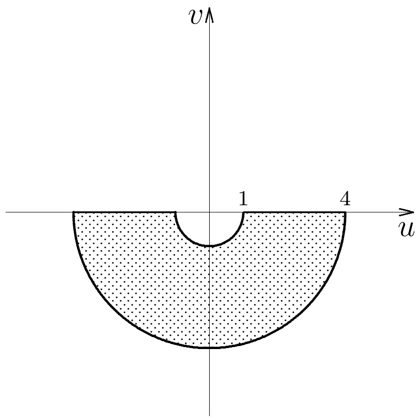

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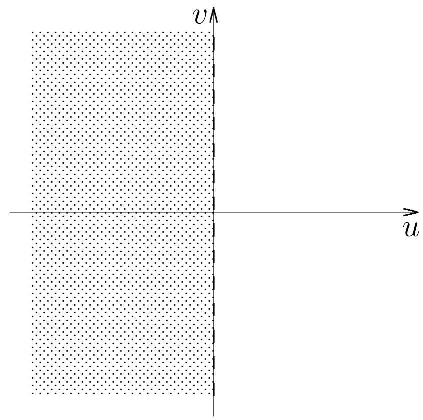
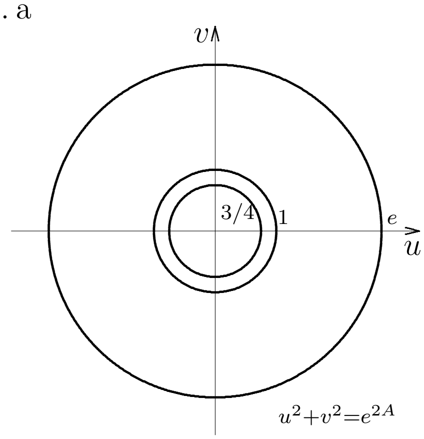
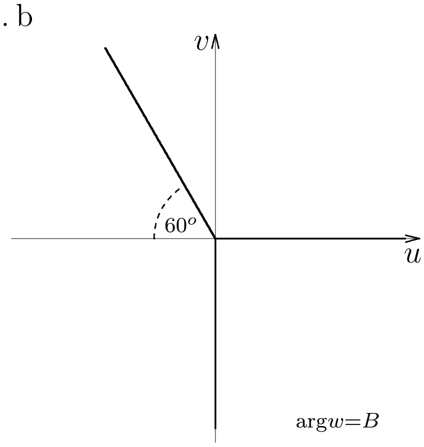


12. a



12. b



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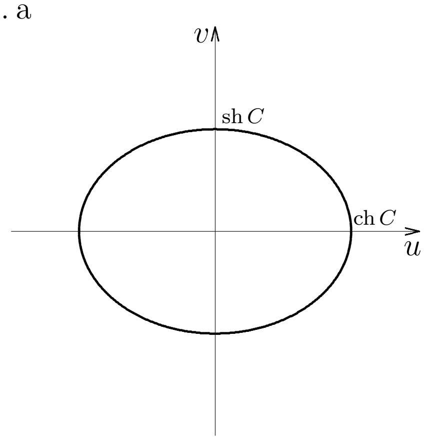
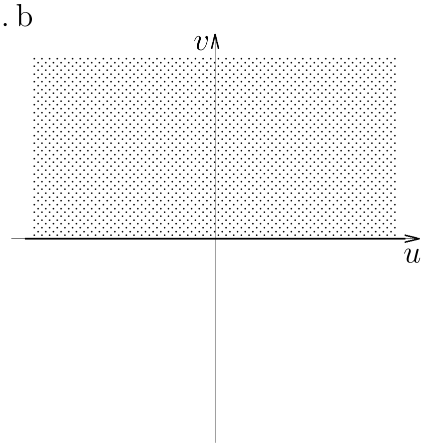
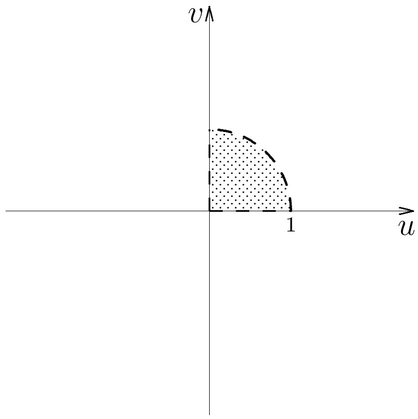


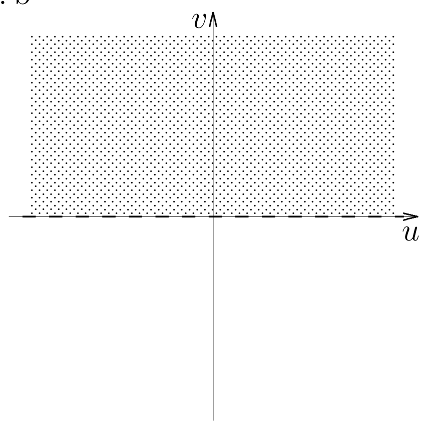
13. b



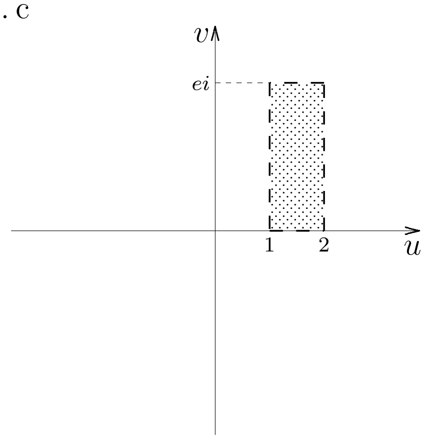
14. a



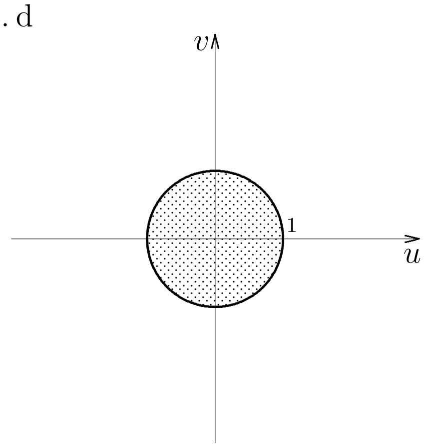
14. b



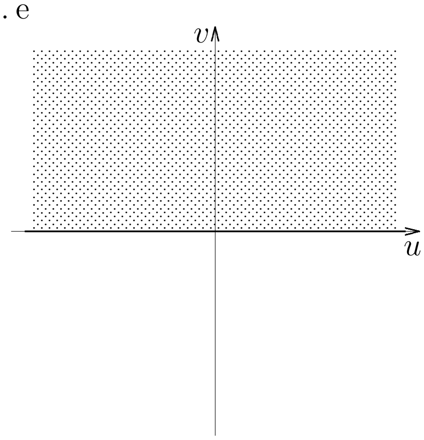
14. c



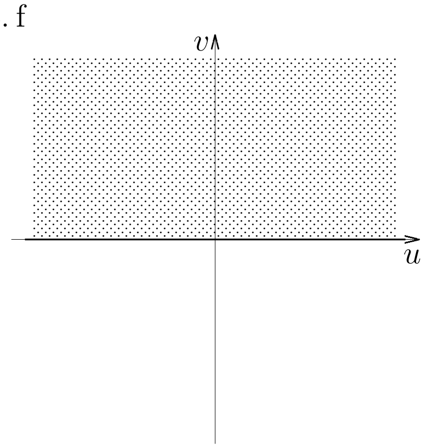
14. d



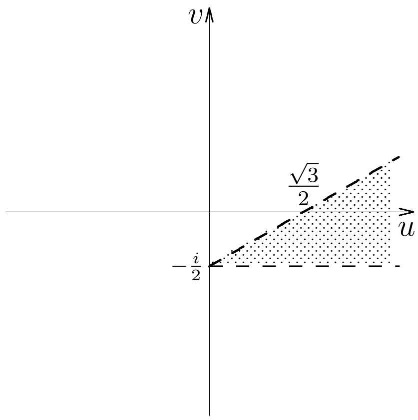
14. e



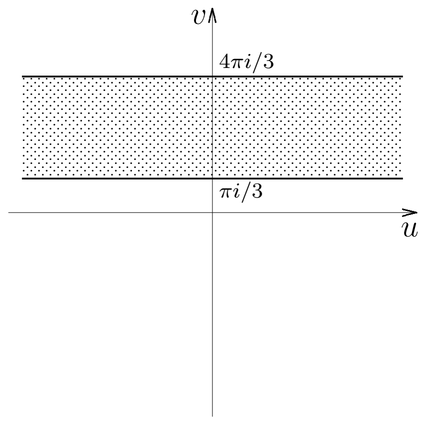
14. f



14. g



14. h



15. $w = -(z - 1 + i)^2$

16. $w = -iz^2$