# IZPIT IZ MATEMATIKE III

## 22. junij 2009

1. Vzemimo skalarno polje $F(x,y,z) = xz \arcsin\left(\frac{y}{3}\right) + xz^2 + xe^y$ in točko $T(1,0,3)$.

   (a) Izračunajte smerni odvod skalarnega polja $F$ v točki $T$ in smeri največjega spreminjanja polja.

   (b) Poščite nivojsko ploskev skalarnega polja $F$, ki gre skozi točko $T$ in nato v tej točki izračunajte še tangentno ravnino na to nivojsko ploskev.

   **Rešitev.**

   (a) Vemo, da je smer najhitrejšega spreminjanja ravno smer gradienta. Tako dobimo

   $$\operatorname{grad} F\,(x,y,z) = (F_x, F_y, F_z) =$$
   $$= \left( z \arcsin\left(\frac{y}{3}\right) + z^2 + e^y, \frac{xz}{3\sqrt{1-\frac{y^2}{9}}} + xe^y, x \arcsin\left(\frac{y}{3}\right) + 2xz \right)$$
   $$\operatorname{grad} F\,(1,0,3) = (10,2,6)$$

   in zato se iskani smerni odvod glasi

   $$\operatorname{grad} F\,(1,0,3) \cdot \frac{\operatorname{grad} F\,(1,0,3)}{|\operatorname{grad} F\,(1,0,3)|} = ... = \sqrt{140} = 2\sqrt{35}.$$

   (b) Nivojske ploskve imajo obliko $F(x,y,z) = C$. Ker velja $F(1,0,3) = 10$, se iskana nivojska ploskev glasi

   $$xz \arcsin\left(\frac{y}{3}\right) + xz^2 + xe^y = 10.$$

   Za tangentno ravnino implicitno podane ploskve vemo, da normalo poračunamo kot $\vec{n}(x,y,z) = (F_x, F_y, F_z)$, torej velja, da je normala v točki $T$ enaka $(10,2,6) \sim (5,1,3)$. Tangentna ravnina se tako glasi $5x + y + 3z = d$, kjer $d$ določimo z vstavljanjem točke $T$ v ravnino, kar nam da $d = 14$. Iskana enačba ravnine je tako

   $$5x + y + 3z = 14.$$

2. Izračunajte

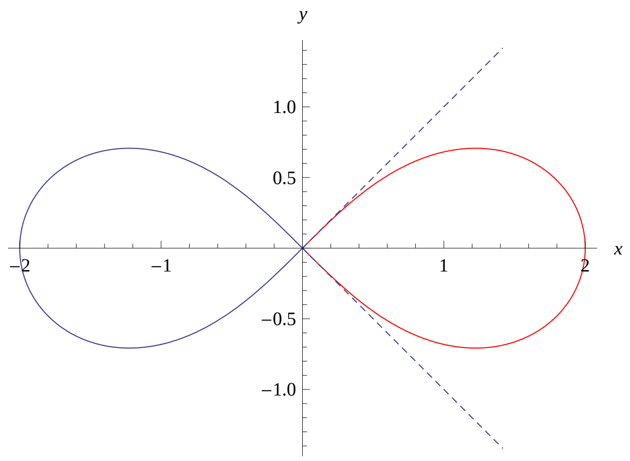
$$\iint_D (1 + x^2 + y^2)\, dx\, dy,$$

kjer območje $D$ predstavlja desno zanko lemniskate

$$(x^2 + y^2)^2 = 4(x^2 - y^2).$$

*Navodilo:* Uvedite polarne koordinate!

**Rešitev.** V polarnih koordinatah se naša lemniskata glasi $r^4 = 4r^2 \cos(2\varphi)$ oziroma $r = 2\sqrt{\cos(2\varphi)}$. Iz definicijskega območja korena vključno z zahtevo po desni zanki dobimo, da mora veljati $-\frac{\pi}{4} \leq \varphi \leq \frac{\pi}{4}$.




Naš integral se tako prevede do:

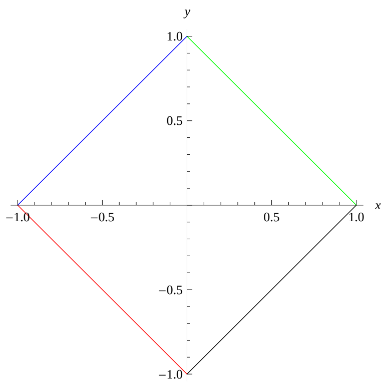
$$\begin{aligned}
\ldots &= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} d\varphi \int_0^{2\sqrt{\cos(2\varphi)}} (1 + r^2) r dr = \\
&= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\frac{r^2}{2} + \frac{r^4}{4}\right)\Big|_0^{2\sqrt{\cos(2\varphi)}} d\varphi = \\
&= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \big(2\cos(2\varphi) + 4\cos^2(2\varphi)\big) d\varphi = \\
&= \int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \big(2\cos(2\varphi) + 2 + 2\cos(4\varphi)\big) d\varphi = \\
&= \left(\sin(2\varphi) + 2\varphi + \frac{\sin(4\varphi)}{2}\right)\Big|_{-\frac{\pi}{4}}^{\frac{\pi}{4}} = \\
&= 1 + 1 + \frac{\pi}{2} + \frac{\pi}{2} + 0 + 0 = 2 + \pi.
\end{aligned}$$

3. Izračunajte krivuljni integral

$$\int_C 5x^2y^2\,ds,$$

kjer je krivulja $C$ podana z $|x| + |y| = 1$.

**Rešitev.** Krivulja $C$ moramo razbiti na štiri manjše krivulje




in tako dobimo štiri parametrizacije (poimenovane po kvadrantih):

$$\begin{aligned}
C_1: \quad & x = t & \dot{x} = 1\\
& y = 1 - t & \dot{y} = -1\\
& 0 \le t \le 1 &\\
& \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{2} &\\
C_2: \quad & x = t & \dot{x} = 1\\
& y = 1 + t & \dot{y} = 1\\
& -1 \le t \le 0 &\\
& \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{2} &\\
C_3: \quad & x = t & \dot{x} = 1\\
& y = -1 - t & \dot{y} = -1\\
& -1 \le t \le 0 &\\
& \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{2} &\\
C_4: \quad & x = t & \dot{x} = 1\\
& y = -1 + t & \dot{y} = 1\\
& 0 \le t \le 1 &\\
& \sqrt{\dot{x}^2 + \dot{y}^2} = \sqrt{2} &
\end{aligned}$$

Tako dobimo:

$$\begin{aligned}\int_C 5x^2y^2\,ds =& 5\int_0^1 t^2(1-t)^2\sqrt{2}\,dt + 5\int_{-1}^0 t^2(1+t)^2\sqrt{2}\,dt +\\ &+ 5\int_{-1}^0 t^2(-1-t)^2\sqrt{2}\,dt + 5\int_0^1 t^2(-1+t)^2\sqrt{2}\,dt =\\ &= 5\sqrt{2}\Bigg(\Big(\frac{t^3}{3}-\frac{t^4}{2}+\frac{t^5}{5}\Big)\Big|_0^1 + \Big(\frac{t^3}{3}+\frac{t^4}{2}+\frac{t^5}{5}\Big)\Big|_{-1}^0 +\\ &+ \Big(\frac{t^3}{3}+\frac{t^4}{2}+\frac{t^5}{5}\Big)\Big|_{-1}^0 + \Big(\frac{t^3}{3}-\frac{t^4}{2}+\frac{t^5}{5}\Big)\Big|_0^1\Bigg) =\\ &= 5\sqrt{2}\Bigg(\Big(\frac{1}{3}-\frac{1}{2}+\frac{1}{5}\Big) + \Big(\frac{1}{3}-\frac{1}{2}+\frac{1}{5}\Big) + \Big(\frac{1}{3}-\frac{1}{2}+\frac{1}{5}\Big) +\\ &+ \Big(\frac{1}{3}-\frac{1}{2}+\frac{1}{5}\Big)\Bigg) =\\ &= \frac{2\sqrt{2}}{3}\end{aligned}$$

4. Izračunajte

$$\iint_S \big(x^2+3x+2y\arctan x\big)dy\,dz + \Big(4y-\frac{y^2}{1+x^2}\Big)dx\,dz + \big(2xz-2z\big)dx\,dy,$$

kjer je $S$ zunanja stran piramide, omejene z

$$x=0,\ y=0,\ z=0,\ x+y+z=1.$$

*Namig:* Gaussov izrek.

**Rešitev.** Sledimo namigu:

$$\begin{aligned}&\operatorname{div}\Big(x^2+3x+2y\arctan x, 4y-\frac{y^2}{1+x^2}, 2xz-2z\Big) =\\ &= 2x+3+\frac{2y}{1+x^2}+4-\frac{2y}{1+x^2}+2x-2 =\\ &= 4x+5\end{aligned}$$

in zato se po Gaussovem izreku integral preoblikuje do

$$
\begin{aligned}
... &= \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} (4x+5)\, dz = \\
&= ... = \int_0^1 dx \int_0^{1-x} (4x+5)(1-x-y)\, dy = \\
&= \int_0^1 \left( \left(5y - xy - 4x^2y - \frac{5y^2}{2} - 2xy^2\right)\Big|_0^{1-x} \right) dx = \\
&= \int_0^1 \left(\frac{5}{2} - 3x - \frac{3x^2}{2} + 2x^3\right) dx = \\
&= ... = \left(\frac{5x}{2} - \frac{3x^2}{2} - \frac{x^3}{2} + \frac{x^4}{2}\right)\Big|_0^1 = \\
&= 1
\end{aligned}
$$

5. Izračunajte kompleksni integral

$$\int_{|z+1|=2} \frac{10}{z^2(z^2+1)(z-2)}\, dz,$$

kjer je integracija v pozitivni smeri.

**Rešitev.** Integracijska krivulja je krožnica s središčem v točki $S(-1,0)$ in polmerom 2, zato so singularnosti znotraj krivulje: $0, i, -i$ (singularnost $z=2$ je pa zunaj!). Poračunajmo si torej relevantne residuume:

$$
\begin{aligned}
\operatorname{Res}_{z=0} \frac{10}{z^2(z^2+1)(z-2)} &= \lim_{z\to 0} \left(\frac{10}{(z^2+1)(z-2)}\right)' = \\
&= \frac{-10\big(2z(z-2)+z^2+1\big)}{(z^2+1)^2(z-2)^2} = \\
&= \frac{-10(3z^2-4z+1)}{(z^2+1)^2(z-2)^2} = -\frac{10}{4} = -\frac{5}{2} \\
\operatorname{Res}_{z=i} \frac{10}{z^2(z^2+1)(z-2)} &= \lim_{z\to i} \frac{10}{z^2(z+i)(z-2)} = \\
&= \frac{10}{-2i(i-2)} = \frac{10(2-4i)}{4+16} = 1-2i \\
\operatorname{Res}_{z=-i} \frac{10}{z^2(z^2+1)(z-2)} &= \overline{\operatorname{Res}_{z=i} \frac{10}{z^2(z^2+1)(z-2)}} = \\
&= \overline{1-2i} = 1+2i
\end{aligned}
$$

Dobimo, da velja:

$$\begin{aligned}\int_{|z+1|=2} \frac{10}{z^2(z^2+1)(z-2)}\,dz = 2\pi i\Bigg(&\operatorname{Res}_{z=0}\frac{10}{z^2(z^2+1)(z-2)}+\\ &+\operatorname{Res}_{z=i}\frac{10}{z^2(z^2+1)(z-2)}+\\ &+\operatorname{Res}_{z=-i}\frac{10}{z^2(z^2+1)(z-2)}\Bigg)=\\ &= 2\pi i\Big(-\frac{5}{2}+1-2i+1+2i\Big) = -\pi i\end{aligned}$$