2. Kolokvij Matematika III
15. januar 2004
Rešitve

1. naloga

a)

Enačba *nivojske* ploskve bo $F(x,y,z) = F(1,2,0)$, torej $\boxed{zx + e^{y\arcsin\frac{z}{2}} = 1}$

b)

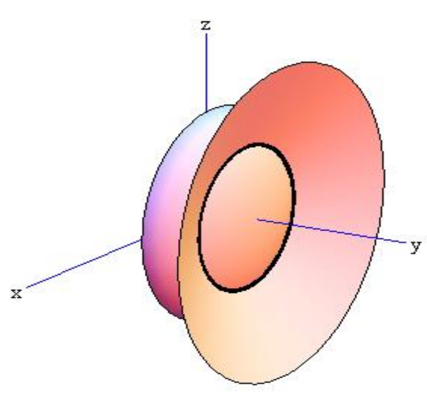
$$\operatorname{grad} F = \left(z\,,\, e^{y\arcsin\frac{z}{2}}\arcsin\tfrac{z}{2}\,,\, x + \frac{y e^{y\arcsin\frac{z}{2}}}{\sqrt{4-z^2}}\right) \quad , \quad \operatorname{grad} F(T) = (0,0,2)$$

Smer najhitrejšega spreminjanja je smer gradienta: $\vec{l} = (0,0,1)$

$$\frac{\partial F}{\partial l} = \operatorname{grad} F(T)\cdot \vec{l} = (0,0,2)\cdot(0,0,1) = \boxed{2}$$

c)

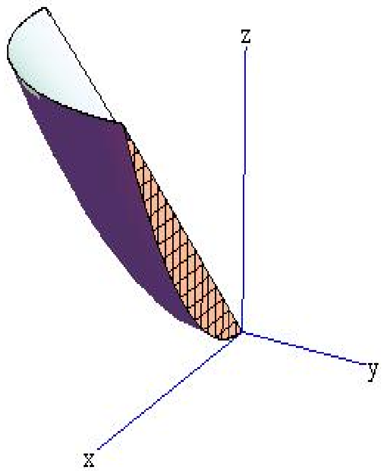
Če enačbi ploskev enkrat seštejemo in drugič odštejemo vidimo, da je krivulja C krožnica $x^2 + z^2 = 4$ v ravnini $y = +1$; za njeno parametrično enačbo lahko vzamemo $\vec{r} = (2\cos t, 1, 2\sin t)$ in $ds = \sqrt{4\sin^2 t + 0 + 4\cos^2 t}dt = 2dt$

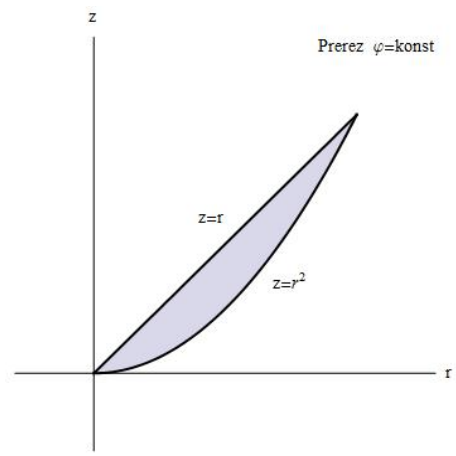


$$\oint_C F(x,y,z)ds = \int_0^{2\pi}\left(2\cos t\, 2\sin t + e^{1\arcsin(\sin t)}\right)2dt = \int_0^{2\pi}(2\sin 2t + e^t)2dt =$$

$$2(-\cos 2t + e^t)\Big|_0^{2\pi} = \boxed{2(e^{2\pi} - 1)}$$

2. naloga





Uporabimo *Gaussovo* formulo in za trojni integral *cilindrične* koordinate.

$$\iint\limits_{rob} \vec{V} \cdot d\vec{S} = \iiint\limits_{območje} \operatorname{div} \vec{V} dV = \iiint\limits_{območje} (-48xyz - 21x^2y - 21y^3)\, dxdydz =$$

$$\int_0^1 r\, dr \int_{r^2}^{r} dz \int_{-\frac{\pi}{2}}^{0} (-48r^2 z \cos\varphi \sin\varphi - 21r^3 \sin\varphi)\, d\varphi \;=$$

$$\int_0^1 r\, dr \int_{r^2}^{r} dz \left(-24r^2 z \sin^2\varphi + 21r^3 \cos\varphi\right)\Bigg|_{-\frac{\pi}{2}}^{0} = \int_0^1 r\, dr \int_{r^2}^{r} dz\, (24r^2 z + 21r^3) =$$

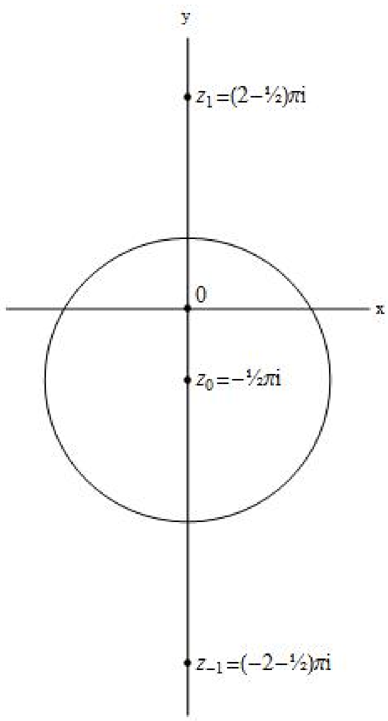
$$\int_0^1 r\, dr \left(12r^2 z^2 + 21r^3 z\right)\Bigg|_{r^2}^{r} = \int_0^1 \left(33r^5 - 12r^7 - 21r^6\right) dr \;= \frac{11}{2} - \frac{3}{2} - 3 = \boxed{1}$$

3. naloga

a)

$$e^z = -i \quad \rightarrow \quad z = \ln(-i) = \ln|-i| + i\arg(-i) \quad , \quad \boxed{z_n = (2n - \frac{1}{2})\pi i}$$

b)



z_n je ničla funkcije $w(z) = e^z + i$. In sicer je ničla 1. stopnje, saj je $w'(z_n) = e^{z_n} = -i \neq 0$.

Znotraj integracijske krivulje C sta dve singularni točki:

0 je pol 2.stopnje

$z_0 = -\frac{1}{2}\pi i$ je pol 1.stopnje.

$$\operatorname*{res}_{z=0} = \lim_{z\to 0}\Big(\frac{1}{e^z + i}\Big)' = \lim_{z\to 0}\Big(-\frac{e^z}{(e^z+i)^2}\Big) = -\frac{1}{(1+i)^2} = \frac{i}{2}$$

$$\operatorname*{res}_{z=-\frac{\pi}{2}i} = \lim_{z\to -\frac{\pi}{2}i} \frac{z + \frac{\pi}{2}i}{z^2(e^z+i)} =$$

Uporabimo pravilo za limito produkta in *l'Hospitalovo* pravilo

$$= \lim_{z\to -\frac{\pi}{2}i} \frac{1}{z^2} \cdot \lim_{z\to -\frac{\pi}{2}i} \frac{1}{e^z} = -\frac{4}{\pi^2} i$$

$$\oint_C \frac{1}{z^2(e^z+i)} dz = 2\pi i\Big(\frac{i}{2} - \frac{4}{\pi^2} i\Big) = \boxed{\frac{8}{\pi} - \pi}$$