2. Kolokvij Matematika III
4. januar 2005
Rešitve

1. naloga

$$\operatorname{rot}\vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{ay}{1+x^2} & e^{yz}z + \operatorname{arctg} x & e^{yz}y + (a-1)x^2 \end{vmatrix} =$$

$$\left(e^{yz}zy + e^{yz} - e^{yz}yz - e^{yz} \quad , \quad -(a-1)2x \quad , \quad \frac{1}{1+x^2} - \frac{a}{1+x^2} \right)$$

Polje bo *potencialno*, če bo $\operatorname{rot}\vec{V} = 0$. Rešitev je

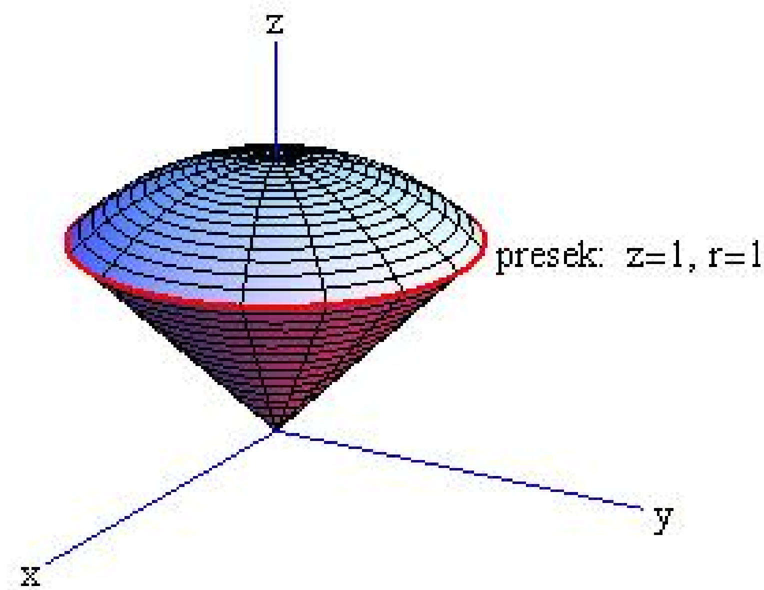
$$\boxed{a = 1}$$

$$u = \int \frac{y}{1+x^2}dx = y \operatorname{arctg} x + C_1(y,z)$$
$$u = \int (e^{yz}z + \operatorname{arctg} x)dy = e^{yz} + y \operatorname{arctg} x + C_2(x,z)$$
$$u = \int e^{yz}y dz = e^{yz} + C_3(x,y)$$

Vsem trem rešitvam ustreza *potencial*

$$\boxed{u = e^{yz} + y \operatorname{arctg} x + C}$$

2. naloga



Uporabili bomo *Gaussovo* formulo in račun izvedli v *cilindričnih* koordinatah. Enačbi robnih ploskev telesa v teh koordinatah sta $r^2 + z^2 = 2$ in $z = r$. Presek ploskev je pri $z = 1$, $r = 1$.

$$\iint_S x^2 dydz + xzdxdz + z^2 dxdy = \iint_S \left(x^2\,,\, xz\,,\, z^2\right) \cdot d\vec{S} = \iiint_{kornet} \mathrm{div}\left(x^2\,,\, xz\,,\, z^2\right) dV =$$

$$\iiint_{kornet} (2x + 2z)\, dV = \iiint_{kornet} 2x\, dV + \iiint_{kornet} 2z\, dV = \iiint_{kornet} 2z\, dV =$$

$$\int_0^{2\pi} d\varphi \int_0^1 r\, dr \int_r^{\sqrt{2-r^2}} 2z\, dz = 2\pi \int_0^1 r\, dr\, z^2 \Big|_r^{\sqrt{2-r^2}} = 2\pi \int_0^1 \left(2r - 2r^3\right) dr = 2\pi \left(r^2 - \frac{r^4}{2}\right)\Big|_0^1 = \boxed{\pi}$$

3. naloga

(a)

$$f(z) = x - iy + \frac{1}{x+iy} = x - iy + \frac{x-iy}{x^2+y^2}$$

$$u = x + \frac{x}{x^2+y^2}\ ,\ v = -y - \frac{y}{x^2+y^2}$$

Preverimo, ali velja *Cauchy-Riemannova* enačba $u_x = v_y$:

$$1 + \frac{x^2+y^2-x2x}{(x^2+y^2)^2} = -1 - \frac{x^2+y^2-y2y}{(x^2+y^2)^2}$$
$$\frac{x^2+y^2-x2x}{(x^2+y^2)^2} + \frac{x^2+y^2-y2y}{(x^2+y^2)^2} = -2$$

$0 = -2$ je protislovje in zato $\boxed{f(z) \text{ ni analitična}}$

(b)

$$\int\limits_{|z-i|=3} (f(z) - \bar{z})\, dz \ = \int\limits_{|z-i|=3} \frac{1}{z}\, dz \ = 2\pi i \operatorname*{res}_{z=0} \frac{1}{z} = \boxed{2\pi i}$$

(c)

$$\int\limits_{|z|=1} f(z)\, dz \ = \int\limits_{|z|=1} \bar{z}\, dz \ + \int\limits_{|z|=1} \frac{1}{z}\, dz$$

Prvi integral izračunamo s parametrično enačbo enotne krožnice $z = e^{i\varphi}$

$$\int\limits_{|z|=1} \bar{z}\, dz \ = \int\limits_0^{2\pi} e^{-i\varphi}\, e^{i\varphi} i d\varphi = 2\pi i$$

Drugi integral je enak integralu pod vprašanjem **(b)**. Rezultat je

$$\int\limits_{|z|=1} f(z)\, dz \ = \boxed{4\pi i}$$