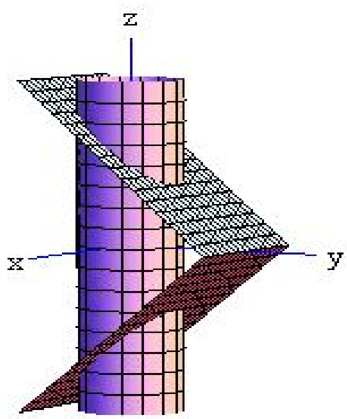
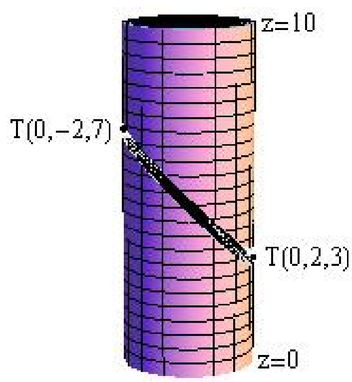
2. Kolokvij Matematika III
11. januar 2006
Rešitve

1. naloga





$$\iint\limits_{rob} \vec{V} \cdot d\vec{S} = \iiint\limits_{območje} \operatorname{div} \vec{V} dV = \iiint\limits_{območje} (2+2z)\, dxdydz =$$

Vpeljemo cilindrične koordinate.

$$\int_0^{2\pi} d\varphi \int_0^2 r\ dr \int_{r\sin\varphi-5}^{5-r\sin\varphi} (2+2z)dz = \int_0^{2\pi} d\varphi \int_0^2 r\ dr \left(2z \Big|_{r\sin\varphi-5}^{5-r\sin\varphi} + z^2 \Big|_{r\sin\varphi-5}^{5-r\sin\varphi} \right) =$$

$$\int_0^{2\pi} d\varphi \int_0^2 r\ dr \left(2\cdot 2z \Big|_0^{5-r\sin\varphi} + 0 \right) = \int_0^{2\pi} d\varphi \int_0^2 r\ dr (20 - 4r\sin\varphi) =$$

$$\int_0^{2\pi} d\varphi \left(10r^2 - \frac{4}{3} r^3 \sin\varphi \right)\Big|_0^2 = \int_0^{2\pi} d\varphi\, (40 - \frac{32}{3}\sin\varphi) = 40\varphi + \frac{32}{3}\cos\varphi \Big|_0^{2\pi} = \boxed{80\pi}$$

Drugi rešitev

Kot je videti iz zgornje rešitve, člen $2z$ v divergenci ne prinese nič k rezultatu. To se da sklepati vnaprej, saj je funkcija $2z$ liha na območju, katerega volumen se drugače zloži (glej drugo sliko).

$$\iint\limits_{rob} \vec{V} \cdot d\vec{S} = \iiint\limits_{območje} 2\, dxdydz = 2 \cdot volumen\ območja = 2\pi 2^2 (3+7) = \boxed{80\pi}$$

2. naloga

(a)

$u_x = 6xy + \cos(2x)2e^{2y}$

$u_{xx} = 6y - \sin(2x)4e^{2y}$

$u_y = 3x^2 + \sin(2x)e^{2y}2 - 3y^2$

$u_{yy} = \sin(2x)e^{2y}4 - 6y$

$\triangle u = u_{xx} + u_{yy} = 0$

u **je** *harmonična* funkcija, zato v obstaja in se ga dobi iz *Cauchy-Riemannovega* sistema.

$v_y = u_x \quad \rightarrow \quad v = \int (6xy + \cos(2x)2e^{2y})dy = 3xy^2 + \cos(2x)e^{2y} + C(x)$

$v_x = -u_y \quad \rightarrow \quad 3y^2 - \sin(2x)2e^{2y} + C'(x) = -3x^2 - \sin(2x)e^{2y}2 + 3y^2 \quad \rightarrow \quad C(x) = -x^3 + K$

$$\boxed{v = 3xy^2 + \cos(2x)e^{2y} - x^3 + K}$$

(b)

$u_x = y + 2^y$

$u_{xx} = 0$

$u_y = x + x2^y \ln 2$

$u_{yy} = x2^y \ln^2 2$

$\triangle u = u_{xx} + u_{yy} = x2^y \ln^2 2$

Ker je $\triangle u \neq 0$, funkcija u **ni** *harmonična*, zato $u + iv$ ne more biti *analitična* funkcija.

3. naloga

$$f(z) = \frac{1}{z(z+1)^3(z+3)}$$

$$\operatorname*{res}_{z=0} f(z) = \lim_{z\to 0} \frac{1}{(z+1)^3(z+3)} = \frac{1}{3}$$

$$\operatorname*{res}_{z=-1} f(z) = \frac{1}{2} \lim_{z\to -1} \Big[\frac{1}{z^2+3z}\Big]'' = \frac{1}{2} \lim_{z\to -1} \Big[\frac{-2z-3}{(z^2+3z)^2}\Big]' =$$

$$\frac{1}{2} \lim_{z\to -1} \frac{-2(z^2+3z)^2 + (2z+3)2(z^2+3z)(2z+3)}{(z^2+3z)^4} = \frac{1}{2}\,\frac{-2\cdot 4 + 2(-2)}{16} = -\frac{3}{8}$$

$$I = 2\pi i\Big(\frac{1}{3} - \frac{3}{8}\Big) = \boxed{-\frac{\pi}{12} i}$$