2. Kolokvij Matematika III
8. januar 2007
Rešitve

1. naloga

a)

$$\operatorname{rot}\vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{1}{2}\log(y^2+z^2)-y & \frac{xy}{y^2+z^2} & \frac{xz}{y^2+z^2} \end{vmatrix} =$$

$$\Big(-\frac{xz2y}{(y^2+z^2)^2}+\frac{xy2z}{(y^2+z^2)^2}\ ,\ \frac{1}{2}\frac{2z}{y^2+z^2}-\frac{z}{y^2+z^2}\ ,\ \frac{y}{y^2+z^2}-\frac{1}{2}\frac{2y}{y^2+z^2}+1\Big) = (0,0,1)$$

Ker je $\operatorname{rot}\vec{V} \neq 0$, je integral odvisen od poti.

b)

Parametrična enačba premice skozi A in B: $\vec{r} = \vec{r}_A + t(\vec{r}_B - \vec{r}_A) = (1, 2+2t, 3+3t)$

$$I = \int_0^1 \Big[[\frac{1}{2}\log((2+2t)^2+(3+3t)^2)-(2+2t)]\cdot 0 + \frac{2+2t}{(2+2t)^2+(3+3t)^2}\cdot 2 + \frac{3+3t}{(2+2t)^2+(3+3t)^2}\cdot 3\Big]\,dt =$$

$$\int_0^1 \frac{4+4t+9+9t}{(2+2t)^2+(3+3t)^2}\,dt = \int_0^1 \frac{13(1+t)}{13(1+t)^2}\,dt = \int_0^1 \frac{dt}{1+t} = \log(1+t)\Big|_0^1 = \boxed{\log 2}$$

b) druga rešitev:

Polje $\vec{V}$ je skoraj potencialno; preveč je samo člen -y v prvi komponenti.

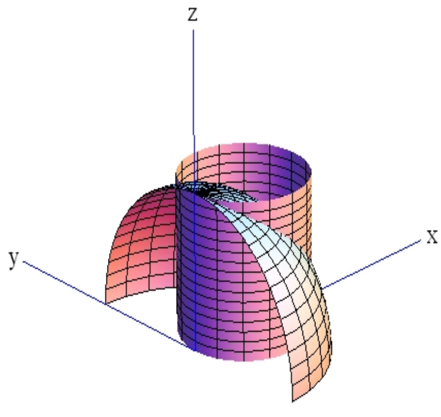
$\vec{V} = \vec{V}_1 + \vec{V}_2$, $\vec{V}_2 = (-y, 0, 0)$, $\operatorname{rot}\vec{V}_1 = 0$, $\vec{V}_1 = \operatorname{grad} u$, $u = \frac{x}{2}\log(y^2+z^2)$

$$\int \vec{V}_1 \cdot d\vec{r} = u(B) - u(A) = \frac{1}{2}(\log 52 - \log 13) = \log 2$$

$$\int \vec{V}_2 \cdot d\vec{r} = -\int y\,dx = -\int_0^1 (2+2t)\cdot 0\,dt = 0$$

$$\int \vec{V}_1 \cdot d\vec{r} + \int \vec{V}_2 \cdot d\vec{r} = \boxed{\log 2}$$

2. naloga



$$\iint\limits_{rob} \vec{V} \cdot d\vec{S} = \iiint\limits_{območje} \operatorname{div} \vec{V} dV = \iiint\limits_{območje} 9\, dxdydz =$$

Vpeljemo cilindrične koordinate in upoštevamo simetrijo območja.

$$2 \int_0^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} r\, dr \int_0^{\sqrt{4-r^2}} 9\, dz \ = 18 \int_0^{\frac{\pi}{2}} d\varphi \int_0^{2\cos\varphi} r\sqrt{4-r^2}\, dr \ = 18 \int_0^{\frac{\pi}{2}} d\varphi \ (-\frac{1}{3}\sqrt{(4-r^2)^3})\Big|_0^{2\cos\varphi} =$$

$$-6 \int_0^{\frac{\pi}{2}} (8\sin^3\varphi - 8)\, d\varphi \ = -48(-\cos\varphi + \frac{1}{3}\cos^3\varphi - \varphi)\Big|_0^{\frac{\pi}{2}} = -48(-\tfrac{\pi}{2} + 1 - \tfrac{1}{3}) = \boxed{24\pi - 32}$$

3. naloga

$$f(z) = \frac{1}{z^2(z^2+1)(z-3)}$$

$$\operatorname*{res}_{z=0} f(z) = \lim_{z\to 0}\Big[\frac{1}{(z^2+1)(z-3)}\Big]' = \lim_{z\to 0} -\frac{2z(z-3)+(z^2+1)}{(z^2+1)^2(z-3)^2} = -\frac{1}{9}$$

$$\operatorname*{res}_{z=i} f(z) = \lim_{z\to i} \frac{1}{z^2(z+i)(z-3)} = \frac{1}{-2i(i-3)} = \frac{1}{2+6i} = \frac{2-6i}{40}$$

$$\operatorname*{res}_{z=-i} f(z) = \lim_{z\to -i} \frac{1}{z^2(z-i)(z-3)} = \frac{1}{2i(-i-3)} = \frac{1}{2-6i} = \frac{2+6i}{40}$$

$$I = 2\pi i\Big(-\frac{1}{9} + \frac{2-6i}{40} + \frac{2+6i}{40}\Big) = 2\pi i\Big(\frac{-1}{9} + \frac{1}{10}\Big) = \boxed{-\frac{\pi}{45} i}$$