DRUGI KOLOKVIJ IZ MATEMATIKE III

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1. (a) Določite konstanto a tako, da bo vektorsko polje

$$\vec{V} = \left((2-a^2)y\sin(2xy),\ \frac{(a+1)^2 z}{1+y^2z^2} - 2x\sin(2xy),\ \frac{y}{1+y^2z^2} - 2z\right)$$

potencialno.

(b) Pri vrednosti parametra a iz prejšnje točke izračunajte integral

$$\int_{T_1}^{T_2} \vec{V}\cdot d\vec{r}$$

od točke $T_1(1,0,2)$ do točke $T_2(0,1,0)$.

Rešitev.

(a) Polje $\vec{V}$ bo potencialno, če bo $\operatorname{rot}\vec{V} = \vec{0}$. Zato računajmo:

$$\begin{aligned}
R_y &= -\frac{2y^2z^2}{(1+y^2z^2)^2} + \frac{1}{1+y^2z^2}\\
Q_z &= -\frac{2(1+a)^2y^2z^2}{(1+y^2z^2)^2} + \frac{(1+a)^2}{1+y^2z^2}\\
P_z &= R_x = 0\\
Q_x &= -4xy\cos(2xy) - 2\sin(2xy)\\
P_y &= -2(2-a^2)xy\cos(2xy) + (2-a^2)\sin(2xy)
\end{aligned}$$

Tako dobimo v resnici le sistem dveh enačb

$$\frac{a(a+2)(y^2z^2-1)}{(1+y^2z^2)^2} = 0\ , \quad (a^2-4)(2xy\cos(2xy)+\sin(2xy)) = 0$$

oziroma

$$a(a+2)=0,\ a^2-4=0 \quad\Longrightarrow\quad a(a+2)=0,\ (a-2)(a+2)=0$$

kar nam da edino rešitev $a=-2$.

(b) Ker je po točki (a) v tem primeru polje $\vec{V}$ potencialno, izračunajmo njegov potencial, saj je vrednost iskanega integrala v takem primeru enaka razliki potencialov v končni in začetni točki

$$U = \int P\,dx = \int (-2y\sin(2xy))\,dx = \cos(2xy) + C(y,z)$$

$$U = \int Q\,dy = \int \left(\frac{z}{1+y^2z^2} - 2x\sin(2xy)\right) dy =$$

$$= \arctan(yz) + \cos(2xy) + C(x,z)$$

$$U = \int R\,dz = \int \left(\frac{y}{1+y^2z^2} - 2z\right) dz = \arctan(yz) - z^2 + C(x,y)$$

$$U = \cos(2xy) + \arctan(yz) - z^2 + C$$

Tako dobimo

$$\int \vec{V}\cdot d\vec{r} = U(T_2) - U(T_1) = 1 - (-3) = 4$$

2. S pomočjo Gaussovega izreka izračunajte pretok vektorskega polja

$$\vec{V} = \left(x^2y + \arctan(yz^2) + e^{xz},\ 4y^3 - xy^2 - yze^{xz},\ 3zy^2 - e^{x^2+y^2}\right)$$

skozi ploskev, ki je rob telesa določenega z neenačbami:

$$z \geq x^2 + y^2 - 2,\quad z \leq 4 - \sqrt{x^2+y^2},\quad y \geq 0.$$

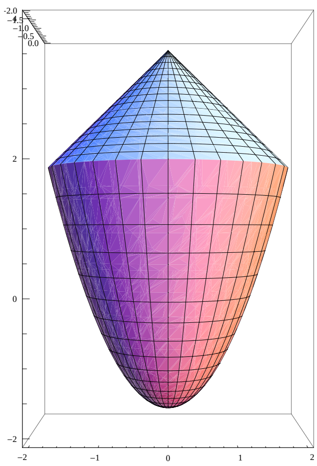
Rešitev. Sledimo navodilu in uporabimo Gaussov izrek.

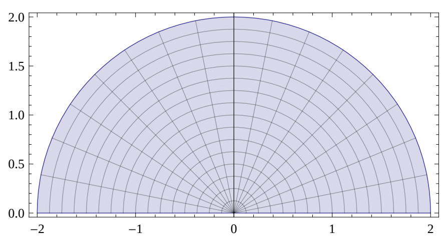
$$\operatorname{div}\vec{V} = \frac{\partial\left(x^2y + \arctan(yz^2) + e^{xz}\right)}{\partial x} + \frac{\partial\left(4y^3 - xy^2 - yze^{xz}\right)}{\partial y} + \frac{\partial\left(3zy^2 - e^{x^2+y^2}\right)}{\partial z} =$$

$$= 2xy + ze^{xz} + 12y^2 - 2xy - ze^{xz} + 3y^2 = 15y^2$$

Poračunamo si presečišče ploskev

$$r^2 - 2 = 4 - r \implies (r+3)(r-2) = 0 \implies r = 2$$





Tako se po uporabi Gaussovega izreka in uvedbi cilindričnih koordinat iskani pretok izračuna

$$\begin{aligned}
\iint_S \vec{V} d\vec{S} &= \iiint_V \operatorname{div} \vec{V} dx\, dy\, dz = \\
&= \int_0^\pi d\varphi \int_0^2 dr \int_{r^2-2}^{4-r} 15r^2 \sin^2(\varphi)\, r\, dz = \\
&= \int_0^\pi d\varphi \int_0^2 \Big(15r^3 \sin^2(\varphi)\, z\Big)\Big|_{r^2-2}^{4-r} dr = \\
&= \int_0^\pi d\varphi \int_0^2 15 \sin^2(\varphi)(6r^3 - r^4 - r^5)\, dr = \\
&= \int_0^\pi \left(15 \sin^2(\varphi)\Big(\frac{3r^4}{2} - \frac{r^5}{5} - \frac{r^6}{6}\Big)\right)\Big|_0^2 d\varphi = \\
&= \int_0^\pi 104 \sin^2(\varphi)\, d\varphi = \\
&= \int_0^\pi 52\big(1 - \cos(2\varphi)\big) d\varphi = \\
&= \big(52\varphi - 26\sin(2\varphi)\big)\Big|_0^\pi = \\
&= 52\pi
\end{aligned}$$

3. (a) Rešite enačbo $\cos z = -3i$.
   (b) Izračunajte kompleksni integral

$$\int_C \frac{dz}{\cos z + 3i},$$

   kjer je C pozitivno orientirana krivulja $\left|z - \left(\frac{\pi}{2} - i\log(\sqrt{10} - 3)\right)\right| = \frac{1}{10}$.

Rešitev. Tekom računanja uvedemo substitucijo $u = e^{iz}$.

(a)

$$\begin{aligned}
\cos z &= -3i \\
\frac{e^{iz} + e^{-iz}}{2} &= -3i \\
u + u^{-1} &= -6i \\
u^2 + 6iu + 1 &= 0 \\
u_{1,2} &= \frac{-6i \pm \sqrt{-36 - 4}}{2} \\
u_{1,2} &= i(-3 \pm \sqrt{10}) \\
e^{iz_{1,2}} &= i(-3 \pm \sqrt{10})
\end{aligned}$$

Tako smo dobili dve možnosti, kjer vsako zase logaritmiramo in dobimo dve družini enačb (kjer $k \in \mathbb{Z}$):

$$\begin{aligned}
iz_{1,k} &= \log \left| i(-3+\sqrt{10}) \right| + i\left(\frac{\pi}{2} + 2k\pi\right) = \log(\sqrt{10}-3) + i\left(\frac{\pi}{2} + 2k\pi\right) \\
iz_{2,k} &= \log \left| i(-3-\sqrt{10}) \right| + i\left(-\frac{\pi}{2} + 2k\pi\right) = \log(\sqrt{10}+3) + i\left(-\frac{\pi}{2} + 2k\pi\right)
\end{aligned}$$

in posledično dve družini rešitev:

$$\begin{aligned}
z_{1,k} &= \left(\frac{\pi}{2} + 2k\pi\right) - i\log(\sqrt{10}-3) \\
z_{2,k} &= \left(-\frac{\pi}{2} + 2k\pi\right) - i\log(\sqrt{10}+3)
\end{aligned}$$

(b) Ker je $z_0 = \frac{\pi}{2} - i\log(\sqrt{10}-3)$ pol prve stopnje za funkcijo $\frac{1}{\cos z + 3i}$, lahko uporabimo formulo:

$$\begin{aligned}
\operatorname{Res}_{z=z_0} \frac{1}{\cos z + 3i} &= \lim_{z \to z_0} \frac{z - \left(\frac{\pi}{2} - i\log(\sqrt{10}-3)\right)}{\cos z + 3i} = \\
&= \lim_{z \to z_0} \frac{1}{-\sin z} = -\frac{1}{\sin\left(\frac{\pi}{2} - i\log(\sqrt{10}-3)\right)} = \\
&= -\frac{2i}{e^{i\left(\frac{\pi}{2} - i\log(\sqrt{10}-3)\right)} - e^{-i\left(\frac{\pi}{2} - i\log(\sqrt{10}-3)\right)}} = \\
&= -\frac{2i}{(\sqrt{10}-3)e^{i\frac{\pi}{2}} - \frac{1}{\sqrt{10}-3}e^{-i\frac{\pi}{2}}} = \\
&= -\frac{2i}{(\sqrt{10}-3)i - (\sqrt{10}+3)(-i)} = \\
&= -\frac{2i}{2\sqrt{10}i} = \\
&= -\frac{\sqrt{10}}{10}
\end{aligned}$$

Tekom računanja smo med drugim uporabili L'Hospital-ovo pravilo, formulo $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$ in dejstvi $e^{\log x} = x$ ter $e^{i\varphi} = \cos\varphi + i\sin\varphi$.
Iskani integral je po izreku o residuih enak

$$\int_C \frac{dz}{\cos z + 3i} = 2\pi i \operatorname{Res}_{z=z_0} \frac{1}{\cos z + 3i} = -2\pi i \frac{\sqrt{10}}{10} = -\pi i \frac{\sqrt{10}}{5}$$