Matematika 4

1. vaja

B. Jurčič Zlobec [1]

[1]Univerza v Ljubljani,
Fakulteta za Elektrotehniko
1000 Ljubljana, Tržaška 25, Slovenija

Matematika FE, Ljubljana, Slovenija 13. marec 2013

Formule: Tomšič & Slivnik

- Prema transformacija

$$F(\omega) = \mathcal{F}(f(t)) = \int_{-\infty}^{\infty} f(t)e^{i\omega t}dt$$

- Obratna transformacija

$$f(t) = \mathcal{F}^{-1}(f(t)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega t}d\omega$$

- Parsevalova enačba

$$\int_{-\infty}^{\infty} |f(t)|^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

Formule: Mathematica©

- Prema transformacija

$$F(\omega) = \mathcal{F}(f(t)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{i\omega t}dt$$

- Obratna transformacija

$$f(t) = \mathcal{F}^{-1}(f(t)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega t}d\omega$$

- Parsevalova enačba

$$\int_{-\infty}^{\infty} |f(t)|^2 = \int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

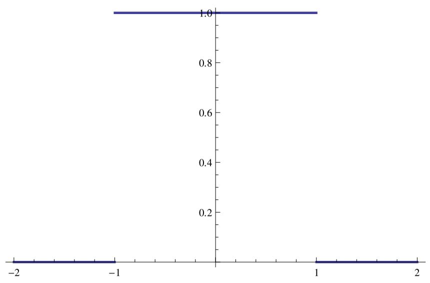
Pravila

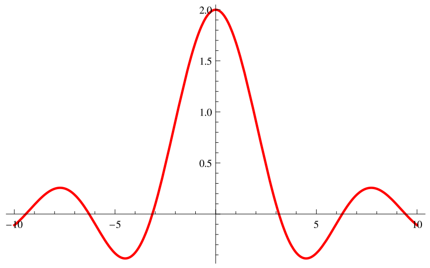
1. Linearnost Fouriereve transformacije: $\mathcal{F}(a\,f(t) + b\,g(t)) = a\,F(\omega) + b\,G(\omega)$.
2. Konvolucija: $\mathcal{F}((f * g)(t)) = F(\omega)G(\omega)$.
3. Premik v t: $\mathcal{F}(f(t - t_0)) = e^{-i\omega t_0}F(\omega)$.
4. Premik v ω: $\mathcal{F}\left(e^{-i\omega_0 t}f(t)\right) = F(\omega - \omega_0)$.
5. Razteg: $\mathcal{F}(f(a\,t)) = \frac{1}{a}F\left(\frac{\omega}{a}\right)$, $a > 0$.
6. Odvod funkcije $f(t)$: $\mathcal{F}\left(f^{(n)}(t)\right) = (-i)^n\omega^n F(\omega)$.
7. Odvod funkcije $F(\omega)$: $\mathcal{F}(t^n f(t)) = (-i)^n F^{(n)}(\omega)$.

Poišči Fourierevo transformacijo funkcije

$$f(t)=\begin{cases}1 & -1\le x\le 1\\ 0 & \text{drugod}\end{cases}.$$

- $\int_{-\infty}^{\infty} e^{i\omega t} f(t)\,dt = \int_{-1}^{1} e^{i\omega t}\,dt \Longrightarrow$
- $= \frac{1}{\omega} e^{i\omega t}\Big|_{-1}^{1} = \frac{e^{i\omega}-e^{-i\omega}}{\omega} = \frac{2\sin\omega}{\omega}.$
- Ker je funkcija $f(t)$ soda, je njena transformiranka realna.

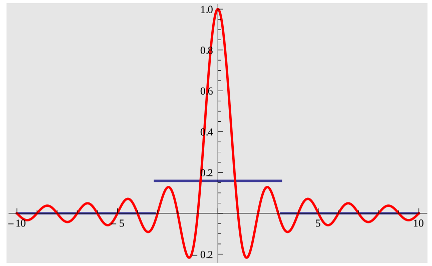


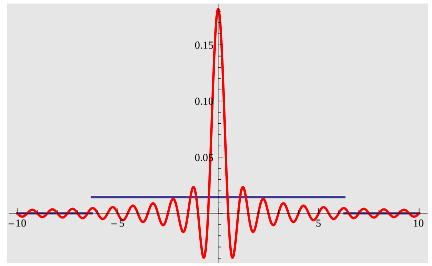


Poišči Fourierevo transformacijo funkcije

$$f(t)=\begin{cases}\frac{1}{2a} & -a\le x\le a\\ 0 & \text{drugod}\end{cases}, \text{ kjer je } a>0.$$

- $\int_{-\infty}^{\infty} e^{i\omega t} f(t)\,dt = \frac{1}{2a}\int_{-a}^{a} e^{i\omega t}\,dt \Longrightarrow$
- $= \frac{1}{2a\omega} e^{i\omega t}\Big|_{-a}^{a} = \frac{e^{i a\omega}-e^{-i a\omega}}{2a\omega} = \frac{\sin(a\omega)}{a\omega}.$
- Večji je a, bolj je $f(t)$ razpršena in bolj je $F(\omega)$ zgoščena in obratno.

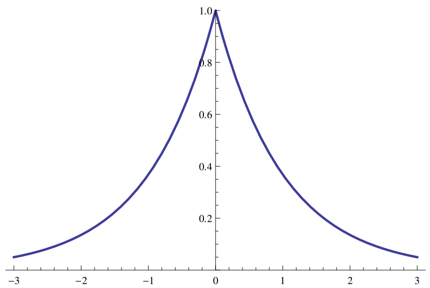


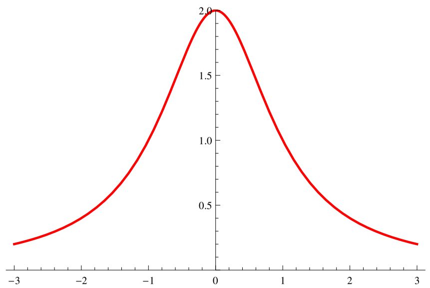


Poišči Fourierevo transformacijo funkcije

$f(t)=e^{-a|t|}$, $a>0$.

- $\int_{-\infty}^{\infty} e^{i\omega t} f(t)\,dt = \int_{-\infty}^{\infty} e^{-a|t|} e^{i\omega t}\,dt \Longrightarrow$
- $= \int_{-\infty}^{0} e^{at} e^{i\omega t}\,dt + \int_{0}^{\infty} e^{-at} e^{i\omega t}\,dt \Longrightarrow$
- $= \frac{e^{t(a+i\omega)}}{a+i\omega}\Big|_{-\infty}^{0} - \frac{e^{-t(a-i\omega)}}{a-i\omega}\Big|_{0}^{\infty} = \frac{2a}{a^2+\omega^2}.$

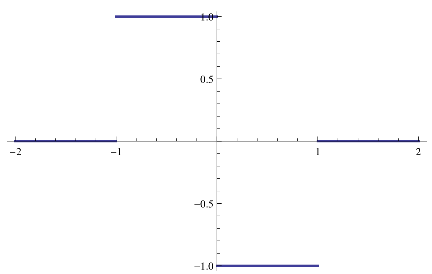


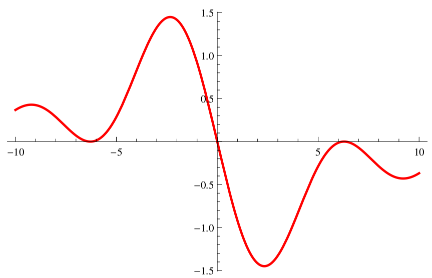


Poišči Fourierevo transformacijo funkcije

$$f(t)=\begin{cases}-1 & -1<t<0\\ 1 & 0\le t<1\\ 0 & \text{drugod}\end{cases}.$$

- $\int_{-\infty}^{\infty} e^{i\omega t} f(t)\,dt = -\int_{-1}^{0} e^{i\omega t}\,dt + \int_{0}^{1} e^{i\omega t}\,dt \Longrightarrow$
- $= \frac{2i(1-\cos\omega)}{\omega}$
- Transformiranka je imaginarna, ker je prvotna funkcija liha.





Dana je funkcija

$f(t) = \begin{cases} e^{-t} & t \geq 0 \\ 0 & t < 0 \end{cases}$. Izračunaj konvolucijo $(f * f)(t)$ na oba načina: z integralom in s pomočjo Fouriereve transformacije.

- $g(t) = \int_{-\infty}^{\infty} f(\tau) f(t-\tau)\, d\tau = \int_0^t e^{-\tau} e^{-t+\tau}\, d\tau = \rightarrow$
- $= e^{-t} \int_0^t d\tau = te^{-t},\ t > 0.$
- $F(\omega) = \mathcal{F}(f(t)) = \int_0^{\infty} e^{-t} e^{i\omega t}\, dt = \frac{i}{i+\omega}.$
- $\mathcal{F}((f * f)(t)) = F^2(\omega) = -\frac{1}{(i+\omega)^2} = (-i)\left(\frac{i}{i+\omega}\right)'.$
- Po formuli 7 je $\mathcal{F}^{-1}(-\frac{1}{(i+\omega)^2}) = te^{-t},\ t > 0.$

Izračunaj integral $\int_0^{\infty} \left(\frac{\sin x}{x}\right)^2 dx.$

Uporabi Parsevalovo enačbo za funkcijo $f(t) = \begin{cases} 1 & -1 \leq t \leq 1 \\ 0 & \text{drugod} \end{cases}$

- $\mathcal{F}(f(t)) = \frac{2 \sin \omega}{\omega}$
- Parsevalova enačba: $\int_{-\infty}^{\infty} |f(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left|\frac{2\sin\omega}{\omega}\right|^2 d\omega.$
- $\int_{-\infty}^{\infty} |f(t)|^2\, dt = 2 = \frac{4}{\pi} \int_0^{\infty} \left|\frac{\sin\omega}{\omega}\right|^2 d\omega \rightarrow$
- $\int_0^{\infty} \left|\frac{\sin\omega}{\omega}\right|^2 d\omega = \frac{\pi}{2}.$

Velja $\mathcal{F}(\frac{1}{\sqrt{2\pi}} e^{\frac{-t^2}{2}}) = e^{\frac{-\omega^2}{2}}.$

- Koliko je $\mathcal{F}(f(t))$, če je $f(t) = \left(\frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-t^2}{2\sigma^2}}\right)$, $\sigma > 0$.
- Uporabimo pravilo 5 in dobimo $\mathcal{F}(f(t)) = e^{-\frac{\sigma^2\omega^2}{2}}.$
- Izračunaj pregib funkcij $(f_1 * f_2)(t)$, kjer je $f_i(t) = \frac{1}{\sqrt{2\pi}\sigma_i} e^{\frac{-t^2}{2\sigma_i^2}}.$
- Fouriereva transformacija pregiba je produkt transformirank. $F(\omega) = \mathcal{F}((f_1 * f_2)(t)) = F_1(\omega) F_2(\omega).$
- $F(\omega) = e^{-\frac{1}{2}(\sigma^2 + \sigma_2^2)\omega^2}$
- $(f_1 * f_2)(t) = \frac{1}{\sqrt{2\pi(\sigma_1^2 + \sigma_2^2)}} e^{-\frac{t^2}{2(\sigma_1^2+\sigma_2^2)}}$

Koliko je Fouriereva transformacija funkcije

$f(t) = \frac{1}{\sqrt{2\pi}} \cos(3t) e^{-\frac{t^2}{2}}$?

- Upoštevamo $\cos(3t) = \frac{1}{2}\left(e^{i3t} + e^{-i3t}\right).$
- Uporabimo pravilo 4. $\mathcal{F}(e^{\pm i3t} g(t)) = G(\omega \pm 3) \rightarrow$ $g(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$ in $G(\omega) = e^{-\frac{\omega^2}{2}}.$
- $F(\omega) = \frac{1}{2}\left(e^{-\frac{1}{2}(\omega+3)^2} + e^{-\frac{1}{2}(\omega-3)^2}\right).$

Slika