Matematika 4

1. vaja

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Formule: Tomšič & Slivnik

- Prema transformacija

$$F(\omega) = \mathcal{F}(f(t)) = \int_{-\infty}^{\infty} f(t)e^{i\omega t}dt$$

- Obratna transformacija

$$f(t) = \mathcal{F}^{-1}(f(t)) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{-i\omega t}d\omega$$

- Parsevalova enačba

$$\int_{-\infty}^{\infty} |f(t)|^2 = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$$

Formule: Mathematica©

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$$F(\omega) = \mathcal{F}(f(t)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt$$

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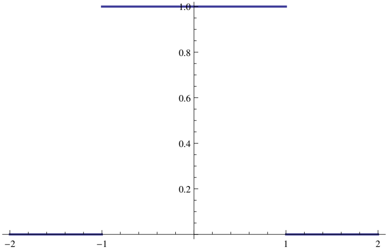
Pravila

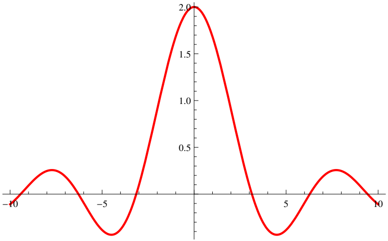
1. Linearnost Fouriereve transformacije: $\mathcal{F}\left(a\,f(t)+b\,g(t)\right)=a\,F(\omega)+b\,G(\omega)$.
2. Konvolucija: $\mathcal{F}\left((f*g)(t)\right)=F(\omega)G(\omega)$.
3. Premik v t: $\mathcal{F}\left(f(t-t_0)\right)=e^{-i\omega t_0}F(\omega)$.
4. Premik v ω: $\mathcal{F}\left(e^{-i\omega_0 t}f(t)\right)=F(\omega-\omega_0)$.
5. Razteg: $\mathcal{F}\left(f(a\,t)\right)=\frac{1}{a}F\left(\frac{\omega}{a}\right)$, $a>0$.
6. Odvod funkcije $f(t)$: $\mathcal{F}\left(f^{(n)}(t)\right)=(-i)^n\omega^nF(\omega)$.
7. Odvod funkcije $F(\omega)$: $\mathcal{F}\left(t^nf(t)\right)=(-i)^nF^{(n)}(\omega)$.

Poišči Fourierevo transformacijo funkcije

$$f(t) = \begin{cases} 1 & -1 \leq x \leq 1 \\ 0 & \text{drugod} \end{cases}.$$

- $\displaystyle\int_{-\infty}^{\infty} e^{i\omega t} f(t)\,dt = \int_{-1}^{1} e^{i\omega t}\,dt = \rightarrow$
- $\displaystyle= \frac{1}{\omega} e^{i\omega t}\Big|_{-1}^{1} = \frac{e^{i\omega} - e^{-i\omega}}{\omega} = \frac{2\sin\omega}{\omega}.$
- Ker je funkcija $f(t)$ soda, je njena transformiranka realna.

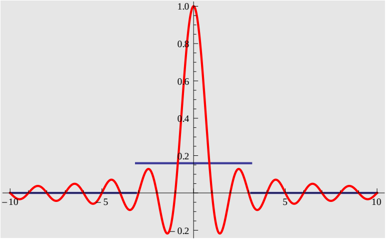


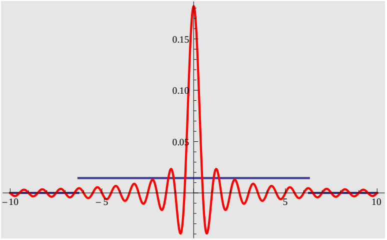


Poišči Fourierevo transformacijo funkcije

$$f(t) = \begin{cases} \frac{1}{2a} & -a \leq x \leq a \\ 0 & \text{drugod} \end{cases}, \text{ kjer je } a > 0.$$

- $\int_{-\infty}^{\infty} e^{i\omega t} f(t)\, dt = \frac{1}{2a} \int_{-a}^{a} e^{i\omega t}\, dt = \rightarrow$
- $= \frac{1}{2a\omega} e^{i\omega t} \Big|_{-a}^{a} = \frac{e^{i\,a\omega} - e^{-i\,a\omega}}{2a\omega} = \frac{\sin(a\omega)}{a\omega}.$
- Večji je a, bolj je $f(t)$ raspršena in bolj je $F(\omega)$ zgoščena in obratno.

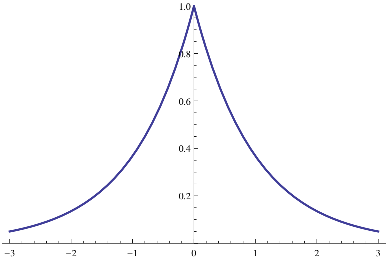


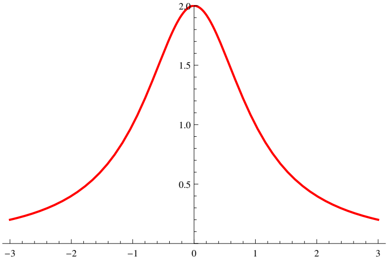


Poišči Fourierevo transformacijo funkcije

$f(t) = e^{-a|t|}$, $a > 0$.

- $\displaystyle\int_{-\infty}^{\infty} e^{i\omega t} f(t)\, dt = \int_{-\infty}^{\infty} e^{-a|t|} e^{i\omega t}\, dt =\rightarrow$
- $\displaystyle= \int_{-\infty}^{0} e^{a\,t} e^{i\omega t}\, dt + \int_{0}^{\infty} e^{-a\,t} e^{i\omega t}\, dt =\rightarrow$
- $\displaystyle= \left.\frac{e^{t(a+i\omega)}}{a+i\omega}\right|_{-\infty}^{0} - \left.\frac{e^{-t(a-i\omega)}}{a-i\omega}\right|_{0}^{\infty} = \frac{2a}{a^2+\omega^2}.$

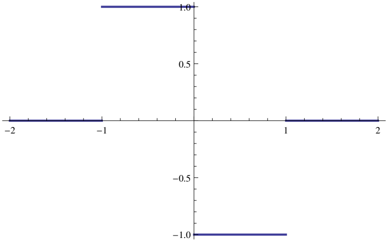


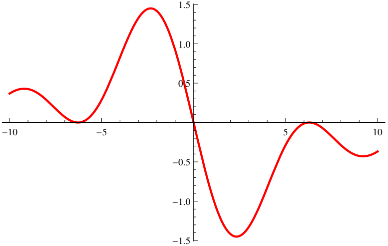


Poišči Fourierevo transformacijo funkcije

$$f(t) = \begin{cases} -1 & -1 < t < 0 \\ 1 & 0 \le t < 1 \\ 0 & \text{drugod} \end{cases}.$$

- $\int_{-\infty}^{\infty} e^{i\omega t} f(t)\, dt = -\int_{-1}^{0} e^{i\omega t}\, dt + \int_{0}^{1} e^{i\omega t}\, dt = \longrightarrow$
- $= \dfrac{2i(1-\cos\omega)}{\omega}$
- Transformiranka je imaginarna, ker je prvotna funkcija liha.





Dana je funkcija

$f(t) = \begin{cases} e^{-t} & t \geq 0 \\ 0 & t < 0 \end{cases}$. Izračunaj konvolucijo $(f * f)(t)$ na oba načina: z integralom in s pomočjo Fouriereve transformacije.

- $g(t) = \int_{-\infty}^{\infty} f(\tau) f(t-\tau)\, d\tau = \int_0^t e^{-\tau} e^{-t+\tau}\, d\tau = \longrightarrow$
- $= e^{-t} \int_0^t d\tau = te^{-t},\ t > 0.$
- $F(\omega) = \mathcal{F}(f(t)) = \int_0^{\infty} e^{-t} e^{i\omega t}\, dt = \dfrac{i}{i+\omega}.$
- $\mathcal{F}\left((f * f)(t)\right) = F^2(\omega) = -\frac{1}{(i+\omega)^2} = (-i)\left(\frac{i}{i+\omega}\right)'.$
- Po formuli 7 je $\mathcal{F}^{-1}(-\frac{1}{(i+\omega)^2}) = te^{-t},\ t > 0.$

Izračunaj integral $\int_0^\infty \left(\frac{\sin x}{x}\right)^2 dx$.

Uporabi Parsevalovo enačbo za funkcijo $f(t) = \begin{cases} 1 & -1 \leq t \leq 1 \\ 0 & \text{drugod} \end{cases}$

- $\mathcal{F}(f(t)) = \frac{2\sin\omega}{\omega}$
- Parsevalova enačba: $\int_{-\infty}^{\infty} |f(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|\frac{2\sin\omega}{\omega}\right|^2 d\omega.$
- $\int_{-\infty}^{\infty} |f(t)|^2\, dt = 2 = \frac{4}{\pi}\int_0^{\infty} \left|\frac{\sin\omega}{\omega}\right|^2 d\omega \to$
- $\int_0^{\infty} \left|\frac{\sin\omega}{\omega}\right|^2 d\omega = \frac{\pi}{2}.$

Velja $\mathcal{F}(\frac{1}{\sqrt{2\pi}}e^{\frac{-t^2}{2}}) = e^{\frac{-\omega^2}{2}}$.

- Koliko je $\mathcal{F}(f(t))$, če je $f(t) = \left(\frac{1}{\sqrt{2\pi}\sigma}e^{\frac{-t^2}{2\sigma^2}}\right)$, $\sigma > 0$.
- Uporabimo pravilo 5 in dobimo $\mathcal{F}(f(t)) = e^{-\frac{\sigma^2\omega^2}{2}}$.
- Izračunaj pregib funkcij $(f_1 * f_2)(t)$, kjer je $f_i(t) = \frac{1}{\sqrt{2\pi}\sigma_i}e^{\frac{-t^2}{2\sigma_i^2}}$.
- Fouriereva transformacija pregiba je produkt transformirank. $F(\omega) = \mathcal{F}((f_1 * f_2)(t)) = F_1(\omega)F_2(\omega)$.
- $F(\omega) = e^{-\frac{1}{2}(\sigma^2+\sigma_2^2)\omega^2}$
- $(f_1 * f_2)(t) = \frac{1}{\sqrt{2\pi(\sigma_1^2+\sigma_2^2)}}e^{-\frac{t^2}{2(\sigma_1^2+\sigma_2^2)}}$

Koliko je Fouriereva transformacija funkcije

$$f(t) = \frac{1}{\sqrt{2\pi}} \cos(3t) e^{-\frac{t^2}{2}}?$$

- Upoštevamo $\cos(3t) = \frac{1}{2}\left(e^{i3t} + e^{-i3t}\right)$.
- Uporabimo pravilo 4. $\mathcal{F}(e^{\pm i3t} g(t)) = G(\omega \pm 3) \rightarrow$ $g(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$ in $G(\omega) = e^{-\frac{\omega^2}{2}}$.
- $F(\omega) = \frac{1}{2}\left(e^{-\frac{1}{2}(\omega+3)^2} + e^{-\frac{1}{2}(\omega-3)^2)}\right)$.

Slika