Matematika 4

3. vaja

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Reši diferencialno enačbo z nastavkom $y(x) = \sum_{i=0}^{\infty} a_i x^i$ in zapiši prvih pet členov vrste različnih od nič.
$y''(x) + x^2 y(x) = 0, \quad y(0) = 1, \quad \text{in} \quad y'(0) = -1/2.$

- $y(x) = \sum_{i=0}^{\infty} a_i x^i$ in $y''(x) = \sum_{i=2}^{\infty} i(i-1)a_i x^{i-2}$.
- $\sum_{i=2}^{\infty} i(i-1)a_i x^{i-2} + \sum_{i=0}^{\infty} a_i x^{i+2} = 0,$
- $(i+1)(i+2)a_{i+2} + a_{i-2} = 0,\ i = 2,3,\dots\ \ a_2 = a_3 = 0,$
- $a_{i+2} = -\dfrac{a_{i-2}}{(i+1)(i+2)},\ i = 2,3,\dots,$
- $y(x) = 1 - \dfrac{x}{2} - \dfrac{x^4}{12} + \dfrac{x^5}{40} + \dfrac{x^8}{672}\dots.$

Reši diferencialno enačbo z nastavkom
$y(x) = x^r \sum_{i=0}^{\infty} a_i x^i$.
$4xy''(x) + 2y'(x) + y(x) = 0,\ y(0) = 1$ in
$\lim_{x\to 0} y'(x) = -\frac{1}{2}$.

- $\sum_{i=0}^{\infty} \left(4(i+r)(i+r-1)a_i + 2a_i(i+r) + a_i x\right) x^{i+r-1} = 0.$
- Prvi pogoj $(4r(r-1) + 2r)a_0 = 0$
- $(4(i+r)(i+r-1) + 2(i+r))a_i + a_{i-1} = 0,\ i = 1,3,\dots$
- $a_i = -\frac{a_{i-1}}{2(i+r)(2i+2r-1)},\ i = 1,3,\dots$
- Če je $a_0 \neq 0$, potem je $r(2r-1) = 0,\ r = 0$ ali $r = \frac{1}{2}$.
- Za $r = 0,\ a_n = -\frac{a_n - 1}{2i(2i-1)} = (-1)^i \frac{a_0}{(2i)!},$
- za $r = \frac{1}{2},\ a_n = (-1)^i \frac{a_0}{(2i+1)!},$
- $a_0^{(1)} \left(1 - \frac{x}{2!} + \frac{x^2}{4!} - \dots\right) + a_0^{(2)} \left(\sqrt{x} - \frac{\sqrt{x}x}{3!} + \frac{\sqrt{x}x^2}{5!} - \dots\right).$
- $y(x) = \cos\sqrt{x}.$

Uvedi novo odvisno spremenljivko v diferencialno enačbo.
$(x^2-1)y''(x) + \left(4x - \frac{2}{x}\right) y'(x) - 10y(x), \quad y = \frac{z}{x}.$

- $y'(x) = \frac{z'(x)}{x} - \frac{z}{x^2},\ y''(x) = \frac{z''(x)}{x} - \frac{2z'(x)}{x^2} + \frac{2z(x)}{x^3}.$
- $(x^2-1)\left(\frac{z''}{x} - \frac{2z'}{x^2} + \frac{2z}{x^3}\right) + \left(4x - \frac{2}{x}\right)\left(\frac{z'}{x} - \frac{z}{x^2}\right) - 10\frac{z}{x}.$
- $(1-x^2)z''(x) - 2xz'(x) + 12z(x) = 0.$

Besslova diferencialna enačba

$$x^2y''(x) + xy'(x) + (x^2 - \nu^2)y(x) = 0$$

- Parameter ν ni celo število, $y(x) = AJ_\nu(x) + BJ_{-\nu}(x)$.
- Parameter $\nu = n$ je celo število, $f(x) = AJ_n(x) + BN_n(x)$.
- Parameter $\nu = \frac{1}{2}$, $y(x) = \sqrt{\frac{2}{\pi x}}(A\sin(x) + B\cos(x))$
- Rekurzivne formule.

$$J_{\nu-1} + J_{\nu+1} = \frac{2\nu}{x}J_\nu(x)$$

- Odvodi Besslovih funkcij.

$$\frac{dJ_\nu(x)}{dx} = -\frac{\nu}{x}J_\nu(x) + J_{\nu-1}(x)$$

Poišči rešitev diferencialne enačbe $x^2y''(x) - xy'(x) + (1+x^2)y(x) = 0$, $y(0) = 0$, $y'(0) = 1$, z uvedbo $y(x) = xz(x)$.

- $y'(x) = z(x) + xz'(x)$, $y''(x) = 2z'(x) + xz''(x)$.
- $x^2(2z'(x) + xz''(x)) - x(z(x) + xz'(x)) + (1+x^2)xz(x) = 0$
- $2xz'(x) + x^2z''(x) - z(x) - xz'(x) + z(x) + x^2z(x) = 0$
- $x^2z''(x) + xz'(x) + x^2z(x) = 0$
- $y(x) = xz(x) = AxJ_0(x) + BxN_0(x)$, $z(0) = 0$, $z'(0) = 1$.
- $y(x) = xJ_0(x)$.

Poišči rešitev diferencialne enačbe $x^2y''(x) + xy'(x) + (4x^4 - 1)y(x) = 0$, $\lim_{x\to 0} y(x) = 0$, $\lim_{x\to 0} y'(x) = 1$, z uvedbo $\xi = x^2$.

- $2x\,dx = d\xi$, $\frac{d\xi}{dx} = 2\sqrt{\xi}$, $\frac{d}{dx} = \frac{d}{d\xi}\frac{d\xi}{dx} = 2\sqrt{\xi}\frac{d}{d\xi}$.
- $y'(x) = 2\sqrt{\xi}\frac{dy}{d\xi} = 2\sqrt{\xi}y_\xi$.
- $y''(x) = 2\sqrt{\xi}\left(2\sqrt{\xi}y_\xi\right)_\xi = 2\sqrt{\xi}\left(\frac{1}{\sqrt{\xi}}y_\xi + 2\sqrt{\xi}y_{\xi\xi}\right) \Longrightarrow$
- $2y_\xi + 4\xi y_{\xi\xi}$, $4\xi^2y_{\xi\xi}(\xi) + 4\xi y_\xi(\xi) + (4\xi^2 - 1)y(\xi) = 0$.
- $\xi^2y_{\xi\xi}(\xi) + \xi y_\xi(\xi) + (\xi^2 - \frac{1}{4})y(\xi) = 0$
- $y(\xi) = AJ_{1/2}(\xi) + BJ_{-1/2}(\xi) = A'\frac{\sin\xi}{\sqrt{\xi}} + B'\frac{\cos\xi}{\sqrt{\xi}}$.
- $y(x) = \frac{1}{x}\sin(x^2)$.

Legendrovi polinomi

- Diferencialna enačba: $(1-x^2)y''(x) - 2xy'(x) = -n(n+1)y(x)$.
- Interval: $[-1,1]$. Utež: $\rho(x) = 1$.
- Rodriguesova formula: $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}\left(x^2-1\right)^n$.
- Ortogonalnost: $(P_m, P_n) = \int_{-1}^{1} P_m(x)P_n(x)dx = \frac{2}{2n+1}\delta_{mn}$.

Poišči rešitev diferencialne enačbe $y''(x) - \tan(x)y'(x) + 2y(x) = 0$, $y(x) = 0$, $y'(x) = 1$, z uvedbo $\xi = \sin(x)$.

- $d\xi = \cos(x)dx$, $\frac{d}{dx} = \frac{d}{d\xi}\frac{d\xi}{dx} = \cos(x)\frac{d}{d\xi}$.
- $y'(x) = \cos(x)\frac{dy}{d\xi}$.
- $y''(x) = \cos(x)\frac{d}{d\xi}\left(\cos(x)\frac{dy}{d\xi}\right)$
- $y''(x) = \cos^2 x\frac{d^2y}{d\xi^2} + \cos(x)\frac{dy}{d\xi}\frac{d\cos(x)}{d\xi}$.
- $\frac{d\cos(x)}{d\xi} = -\sin(x)\frac{dx}{d\xi} = -\tan(x)$
- $y''(x) = \cos^2(x)\frac{d^2y}{d\xi^2} - \sin(x)\frac{dy}{d\xi}$.
- $(1-\xi^2)y''(\xi) - 2\xi y'(\xi) + 2y(\xi) = 0$.
- $y(\xi) = \xi \rightarrow y(x) = \sin(x)$.

Poišči polinomsko rešitev diferencialne enačbe $(x^2-1)y''(x) + \frac{2}{x}y'(x) - (2 + \frac{2}{x^2})y(x) = 0$, z uvedbo $y(x) = xz(x)$.

- $y'(x) = z(x) + xz'(x)$, $y''(x) = 2z'(x) + xz''(x)$.
- $(x^2-1)\left(2z'(x) + xz''(x)\right) + \frac{2}{x}\left(z(x) + xz'(x)\right) - \left(2 + \frac{2}{x^2}\right)xz(x) = 0$.
- $x(x^2-1)z''(x) + (2x^2 - 2 + 2)z'(x) + (\frac{2}{x} - 2x - \frac{2}{x})z(x)$.
- $(1-x^2)z''(x) - 2xz'(x) + 2z(x) = 0$, $2 = l(l+1)$, $l = 1$ od tod $z(x) = P_1(x) = x$ in $y(x) = x^2$.
- $y(x) = x^2$.

Hermitovi polinomi

- Diferencialna enačba: $y''(x) - xy'(x) = -ny(x)$.
- Interval: $(-\infty, \infty)$. Utež: $\rho(x) = e^{-x^2/2}$.
- Rodriguesova formula: $H_n(x) = (-1)^n e^{x^2/2}\dfrac{d^n}{dx^n}e^{-x^2/2}$.
- Ortogonalnost: $(H_m, H_n) = \displaystyle\int_{-\infty}^{\infty} H_m(x)H_n(x)e^{-x^2/2}dx = \sqrt{\pi}n!\delta_{mn}$.

Laguerrovi polinomi

- Diferencialna enačba: $xy''(x) + (1-x)y'(x) = -ny(x)$.
- Interval: $[0, \infty)$. Utež: $\rho(x) = e^{-x}$.
- Rodriguesova formula: $L_n(x) = \dfrac{1}{n!}e^x\dfrac{d^n}{dx^n}\left(x^n e^{-x}\right)$.
- Ortogonalnost: $(L_m, L_n) = \displaystyle\int_0^{\infty} L_m(x)L_n(x)e^{-x}dx = \delta_{mn}$.

Polinomi Čebiševa

- Diferencialna enačba: $(1-x^2)y''(x) - xy'(x) = -n^2 y(x)$.
- Interval: $[-1,1]$. Utež: $\rho(x) = \frac{1}{\sqrt{1-x^2}}$.
- Trigonometrična formula: $T_n(x) = cos(n \arccos(x))$.
- Rekurzivna formula:
  $T_0(x) = 1, \quad T_1(x) = x, \quad T_{n+1} = 2xT_n(x) - T_{n-1}(x)$.
- Ortogonalnost: $(m|n>0), (T_m, T_n) =$
  $\int_{-1}^{1} T_m(x) T_n(x) \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}\delta_{mn}, (T_0, T_0) = \pi$.
- Lastnosti. *Med vsemi polinomi iste stopnje z istim vodilnim koeficientom ima polinom Čebiševa na intervalu* $[-1,1]$ *najmanjše absolutne vrednosti ekstremov.*

Določi konstante α_{ij}, tako, da bodo funkcije:

$f_0(x) = \alpha_{00}$, $f_1(x) = \alpha_{10} + \alpha_{11}x$ in $f_2(x) = \alpha_{20} + \alpha_{21}x + \alpha_{22}x^2$ ortonormirane na intervalu $[-1,1]$.

- $\hat{f}_0(x) = 1$, $\|\hat{f}_0(x)\|^2 = \int_{-1}^{1} \hat{f}_0^2(x)dx = 2$.
- $\hat{f}_1(x) = \alpha\hat{f}_0(x) + x \to$
- $\left(\hat{f}_1(x), \hat{f}_0(x)\right) = \alpha\left(\hat{f}_0(x), \hat{f}_0(x)\right) + \left(x, \hat{f}_0(x)\right) = 0$.
- $\alpha = -\frac{(x,\hat{f}_0(x))}{(\hat{f}_0(x),\hat{f}_0(x))} = -\frac{\int_{-1}^{1} x\,dx}{\int_{-1}^{1} dx} = 0$, $\hat{f}_1(x) = x$, $\|\hat{f}_1(x)\|^2 = \frac{2}{3}$.
- $\hat{f}_2(x) = \alpha_0\hat{f}_0 + \alpha_1\hat{f}_1(x) + x^2$, $\alpha_i = -\frac{(\hat{f}_i(x),x^2)}{(\hat{f}_i(x),\hat{f}_i(x))}$.
- $\hat{f}_2(x) = -\frac{1}{3} + x^2$, $\|\hat{f}_2(x)\|^2 = \frac{8}{45}$. Normirani $\to$
- $f_0(x) = \frac{1}{\sqrt{2}}, \quad f_1(x) = \sqrt{\frac{3}{2}}x, \quad f_2(x) = \sqrt{\frac{5}{8}}(3x^2-1)$.

Določi konstante α_{ij}, tako, da bodo funkcije:

$f_0(x) = \alpha_{00}$, $f_1(x) = \alpha_{10} + \alpha_{11}x$ in $f_2(x) = \alpha_{20} + \alpha_{21}x + \alpha_{22}x^2$ ortonormirane na intervalu $[0,2]$.

- $\hat{f}_0(x) = 1$, $\|\hat{f}_0(x)\|^2 = \int_{0}^{2} \hat{f}_0^2(x)dx = 2$.
- $\hat{f}_1(x) = \alpha\hat{f}_0(x) + x \to$
- $\left(\hat{f}_1(x), \hat{f}_0(x)\right) = \alpha\left(\hat{f}_0(x), \hat{f}_0(x)\right) + \left(x, \hat{f}_0(x)\right) = 0$.
- $\alpha = -\frac{(x,\hat{f}_0(x))}{(\hat{f}_0(x),\hat{f}_0(x))} = -\frac{\int_{0}^{2} x\,dx}{\int_{0}^{2} dx} = -1$, $\hat{f}_1(x) = -1 + x$,
  $\|\hat{f}_1(x)\|^2 = \frac{2}{3}$.
- $\hat{f}_2(x) = \alpha_0\hat{f}_0 + \alpha_1\hat{f}_1(x) + x^2$, $\alpha_i = -\frac{(\hat{f}_i(x),x^2)}{(\hat{f}_i(x),\hat{f}_i(x))}$.
- $\hat{f}_2(x) = \frac{2}{3} - 2x + x^2$, $\|\hat{f}_2(x)\|^2 = \frac{8}{45}$. Normirani $\to$
- $f_0(x) = \frac{1}{\sqrt{2}}, \quad f_1(x) = \sqrt{\frac{3}{2}}(x-1), \quad f_2(x) = \sqrt{\frac{5}{8}}(3(x-1)^2-1)$.

Določi konstante α_{ij}, tako, da bodo funkcije:

$f_0(x) = \alpha_{00}$, $f_1(x) = \alpha_{10} + \alpha_{11}x$ in $f_2(x) = \alpha_{20} + \alpha_{21}x + \alpha_{22}x^2$ ortonormirane na intervalu $[0,2]$, z utežjo $\rho(x) = x$.

- $\hat{f}_0(x) = 1$, $\|\hat{f}_0(x)\|^2 = \int_{0}^{2} x\hat{f}_0^2(x)dx = 2$.
- $\hat{f}_1(x) = \alpha\hat{f}_0(x) + x \to$
- $\left(\hat{f}_1(x), \hat{f}_0(x)\right) = \alpha\left(\hat{f}_0(x), \hat{f}_0(x)\right) + \left(x, \hat{f}_0(x)\right) = 0$.
- $\alpha = -\frac{(x,\hat{f}_0(x))}{(\hat{f}_0(x),\hat{f}_0(x))} = -\frac{\int_{0}^{2} x^2\,dx}{\int_{0}^{2} x\,dx} = -\frac{4}{3}$, $\hat{f}_1(x) = -\frac{4}{3} + x$,
  $\|\hat{f}_1(x)\|^2 = \frac{4}{9}$.
- $\hat{f}_2(x) = \alpha_0\hat{f}_0 + \alpha_1\hat{f}_1(x) + x^2$, $\alpha_i = -\frac{(\hat{f}_i(x),x^2)}{(\hat{f}_i(x),\hat{f}_i(x))}$.
- $\hat{f}_2(x) = \frac{6}{5} - \frac{12}{5}x + x^2$, $\|\hat{f}_2(x)\|^2 = \frac{8}{45}$. Normirani $\to$
- $f_0(x) = \frac{1}{\sqrt{2}}, \quad f_1(x) = -2 + \frac{3x}{2}, \quad f_2(x) = \sqrt{\frac{5}{8}}(6-12x+5x^2)$.

Preizkusi ortogonalnost Lagendrejevih polinomov $P_2(x)$ in $P_3(x)$

- $P_2(x) = \frac{1}{2}\left(3x^2 - 1\right)$,
- $P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right)$.
- $(P_2(x), P_3(x) = \int_{-1}^{1} P_2(x)P_3(x)dx = \rightarrow$
- $\int_{-1}^{1} \frac{15x^5}{4} - \frac{7x^3}{2} + \frac{3x}{4} = 0$.
- $(P_2(x), P_3(x)) = 0$.

Poišči normo Legendrovih polinomov $P_1(x)$ in $P_3(x)$

- $P_1(x) = x \rightarrow$
- $\|P_1(x)\|^2 = \int_{-1}^{1} |P_1(x)|^2 dx = \frac{2}{3}$.
- $P_3(x) = \frac{1}{2}\left(5x^3 - 3x\right) \rightarrow$
- $\|P_3(x)\|^2 = \frac{1}{4}\int_{-1}^{1} \left(5x^3 - 3x\right)^2 dx = \frac{2}{7}$.
- $\|P_1(x)\| = \sqrt{\frac{2}{3}}, \|P_3(x)\| = \sqrt{\frac{2}{7}}$.

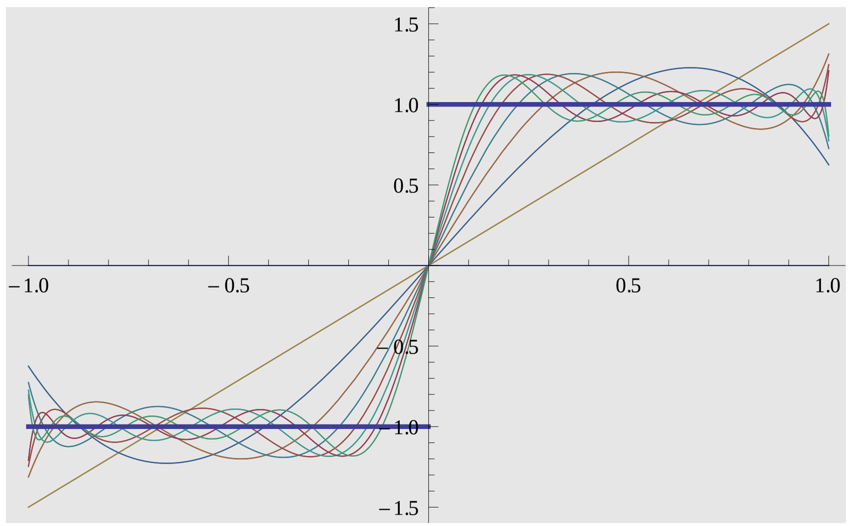
Določi koeficienta a_1 in a_3 v razvoju funkcije $f(x)$

po Legendrovih polinomih na intervalu $[-1, 1]$. Funkcija

$$f(x) = \begin{cases} -1, & -1 \leq x < 0 \\ 1, & 0 < x \leq 1 \\ 0, & x = 0 \end{cases}.$$

- $f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$, za $x \in [-1, 1]$.
- $a_1 = (f(x), P_1(x))/\|P_1(x)\|^2$ in $a_3 = (f(x), P_3(x))/\|P_3(x)\|^2$.
- $a_1 = \frac{3}{2}\int_{-1}^{1} xf(x)dx = \frac{3}{2}\int_{0}^{1} x\,dx = \frac{3}{4}$.
- $a_3 = \frac{7}{2}\int_{-1}^{1} \frac{1}{2}\left(5x^3 - 3x\right) f(x) = \frac{7}{8}\int_{0}^{1} x^3 - 3x\,dx = -\frac{7}{16}$
- $a_1 = \frac{3}{4}, \quad a_3 = -\frac{7}{16}$

Aproksimacija $f(x)$ z Legendrovimi polynomi



Ena izmed rešitev diferencialne enačbe je polinom. Določi njegovo stopnjo.

$y''(x) - xy'(x) + 3y = 0$

- Hermitova diferencialna enačba za $n = 3$.
- $y(x) = H_3(x)$.
- Stopnja je 3.

Poišči Laplaceovo transformacijo Besslove funkcije $J_0(x)$.

- Funkcija $y = J_0(x)$ je rešitev diferencialne enačbe

$$xy''(x) + y'(x) + xy(x) = 0, \quad y(0) = 1 \quad \text{in} \quad y'(0) = 0.$$

- Laplaceova transformacija diferencialne enačbe je enačba

$$-\frac{d}{ds}\left(s^2 Y(s) - s\right) + sX(s) - 1 - Y'(s) = 0,$$

$$Y'(s)(s^2+1) + sY(s) = 0.$$

- Rezultat je diferencialna enačba prvega reda z ločljivimi spremenljivkami.

$$\frac{dY}{Y} = -\frac{s\,ds}{1+s^2} \to \ln Y(s) = -\frac{1}{2}\ln(1+s^2) \to .$$

- $Y(s) = \frac{1}{\sqrt{1+s^2}}$.

Ena izmed rešitev diferencialne enačbe je polinom. Določi njegovo stopnjo.

$y''(x) - 2xy'(x) + 6y = 0$

- $a_{n+2} = 2\frac{(n-3)a_n}{(n+2)(n+1)}$.
- Stopnja je 3.