Matematika 4

5. vaja

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Enačba za nihanje vpete strune

$$\mu\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial}{\partial x}\left(T(x)\,\frac{\partial u(x,t)}{\partial x}\right)$$

Struna je vpeta v $x = 0$ in $x = l$, dolžina je l.
Robni pogoji: $u(0,t) = u(l,t) = 0$.
Napetost strune: $T = konst.$
$c^2 = \frac{T}{\mu}$, kjer je μ dolžinska gostota strune.

- $\frac{1}{c^2}\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial^2 u(x,t)}{\partial x^2}$
- Ločitev spremenljivk $u(x,t) = X(x)T(t)$.
- Lastne funkcije: $-X'' = \lambda^2 X$, $X(x) = \sin(\lambda x)$.
- Lastne vrednosti:
  zaradi robnih pogojev je $\lambda l = n\pi$, $n \in \mathbb{N}$, $\lambda_n = \frac{n\pi}{l}$.
- Časovni del $T'' + \omega_n^2 T = 0$. $\omega_n = c\lambda_n$.
- $T_n(t) = A_n\cos(\omega_n t) + B_n\sin(\omega_n t)$
- $u_n(x,t) = \sin(\lambda_n x)\,(A_n\cos(\omega_n t) + B_n\sin(\omega_n t))$.

Enačba za nihanje krožeče strune

$$\mu\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial}{\partial x}\left(T(x)\,\frac{\partial u(x,t)}{\partial x}\right)$$

Struna dolžine l je vpeta v $x = 0$.
Robni pogoji: $u(0,t) = 0$, $|u(l,t)| < \infty$.
Napetost strune: $dT = x\omega^2\mu\,dx$, $T(x) = \mu\omega^2\int_x^l \xi d\xi$,
$T(x) = \frac{1}{2}\mu\omega^2(l^2 - x^2)$, $c^2 = \frac{1}{2}\omega^2$.

- Uvedemo novo spremenljivko $\xi = \frac{x}{l}$.
- $c^{-2}u_{tt} = \left((1-\xi^2)u_\xi\right)_\xi$.
- Robni pogoji $u(0,t) = 0$, $\quad |u(1,t)| < \infty$.
- Ločitev spremenljivk $u(\xi,t) = \Xi(\xi)T(t)$.
- $-\left((1-\xi^2)\Xi'\right)' = \lambda\Xi$, $(1-\xi^2)\Xi'' - 2\xi\Xi' + \lambda\Xi = 0$.

- Rešitev je lihi Legendreev polinom, ker je $u(0,t) = 0$.
- Lastne vrednosti $\lambda_n = 2n(2n-1)$.
- Lastne funkcije $\Xi_n(\xi) = P_{2n-1}(\xi)$.
- Časovni del: $T'' + c^2\lambda T = 0$, $\omega_n = c\sqrt{\lambda_n} = \omega\sqrt{n(2n-1)}$.
  $T_n(t) = A_n\cos(\omega_n t) + B_n\sin(\omega_n t)$.
- $u_n(x,t) = P_{2n-1}(x/l)\,(A_n\cos(\omega_n t) + B_n\sin(\omega_n t))$.

Enačba za nihanje obešene strune

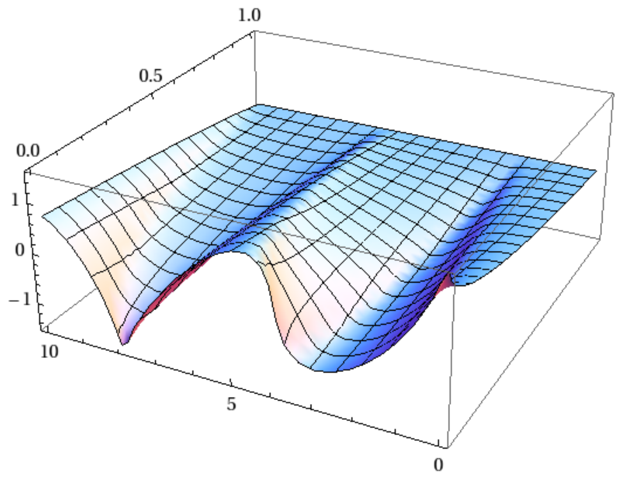
$$\mu\frac{\partial^2 u(x,t)}{\partial t^2} = \frac{\partial}{\partial x}\left(T(x)\,\frac{\partial u(x,t)}{\partial x}\right)$$

Obešena struna, vpeta v $x = l$ in dolžine l. Os x je navpična.
Robni pogoji: $|u(0,t)| < \infty$, $|u(l,t)| = 0$.
Napetost strune: $T(x) = g\mu x$, $c^2 = g$.

- $\dfrac{1}{c^2}\dfrac{\partial^2 u(x,t)}{\partial t^2} = \dfrac{\partial}{\partial x}\left(x\dfrac{\partial u(x,t)}{\partial x}\right)$.
- Uvedemo novo spremenljivko $x = \xi^2$.
- $\left(\dfrac{2}{c}\right)^2 u_{tt} = \dfrac{1}{\xi}\left(\xi u_\xi\right)_\xi$.
- Ločimo spremenljivki $u(\xi,t) = \Xi(\xi)T(t)$, $-\frac{1}{\xi}(\xi\Xi')' = \lambda^2\Xi$.
- $\Xi'' + \frac{1}{\xi}\Xi' + \lambda^2\Xi = 0$.

- Robni pogoji $\Xi(\sqrt{l}) = 0$, $|\Xi(0)| < \infty$.
- Diferencialno enačbo reši Besslova funkcija $\Xi(\xi) = J_0(\lambda\xi)$.
- Lastne vrednosti: $\lambda_n^2 = \xi_{0,n}^2/l$, kjer je $\xi_{0,n}$ n-ta ničla Besslove funkcije $J_0(\xi)$.
- Lastne funkcije: $\Xi_n(\xi) = J_0(\lambda_n\xi)$.
- Časovni del: $T'' + \left(\dfrac{c}{2}\right)^2\lambda^2 T = 0$, $\omega_n = \frac{1}{2}c\lambda_n$.
  $T_n(t) = A_n\cos(\omega_n t) + B_n\sin(\omega_n t)$.
- $u_n(x,t) = J_0(\lambda_n\sqrt{x})\left(A_n\cos(\omega_n t) + B_n\sin(\omega_n t)\right)$.

Nihanje obešene strune



Nihanje vpete okrogle membrane $\Delta u(r,t) = \frac{1}{c^2}\frac{\partial^2 u(r,t)}{\partial t^2}$

Polmer membrane je a. Gledamo lastna nihanja, ki niso odvisna od kota. $\Delta u = u_{rr} + \dfrac{1}{r}u_r$. Robna pogoja $u(a,t) = 0$ in $|u(0,t)| < \infty$.

- Enačba $u_{rr} + \dfrac{1}{r}u_r = \dfrac{1}{c^2}u_{tt}$.
- Ločimo spremenljivke $u(r,t) = R(r)T(t)$.
- $-(R'' + \frac{1}{r}R') = \lambda^2 R$, $T'' + c^2\lambda^2 T = 0$.
- Vpeljemo novo neznanko $\xi = r\lambda$,
  $\dfrac{d^2R(\xi)}{d\xi^2} + \dfrac{1}{\xi}\dfrac{dR(\xi)}{d\xi} + R(\xi) = 0$.
- Ker je $|R(\xi)| < \infty$, je $R(\xi) = J_0(\xi)$, $R(r) = J_0(\lambda r)$.
- Upoštevamo še drugi robni pogoj $R(\lambda a) = 0$, $\lambda a = \xi_{0,n}$,
  n-ta ničla Besslove funkcije $J_0(\xi)$, $\lambda_n = \dfrac{\xi_{0,n}}{a}$.
- $R_n(r) = J_0(\lambda_n r)$, $T'' + c^2\lambda_n^2 T = 0$, $\omega_n = c\lambda_n$.
- $u_n(r,t) = R_n(r)T_n(t) = J_0(\lambda_n r)\left(A_n\cos(\omega_n t) + B_n\sin(\omega_n t)\right)$.

Nihanja pravokotne membrane $\Delta u(x,y,t) = \frac{1}{c^2}\frac{\partial^2 u(r,t)}{\partial t^2}$

Stranici sta a in b. $\Delta u = u_{xx} + u_{yy}$. Robni pogoji $u(0,y,t) = u(a,y,t) = u(x,0,t) = u(x,b,t) = 0$.

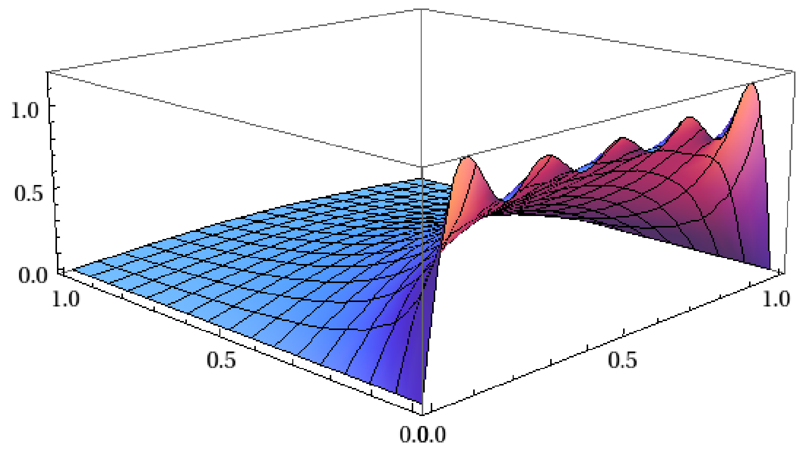
- Enačba $u_{xx} + u_{yy} = \dfrac{1}{c^2} u_{tt}$.
- Ločimo spremenljivke $u(x,y,t) = X(x)Y(y)T(t)$.
- Enačba $\dfrac{X''}{X} + \dfrac{Y''}{Y} = \dfrac{1}{c^2}\dfrac{T''}{T}$, robni pogoji $X(0) = X(a) = Y(0) = Y(b) = 0$.
- Prostorski del: $X'' + \lambda_m^2 X = 0$, $Y'' + \mu_n^2 Y = 0$. Lastni vrednosti $\lambda_m = \frac{m\pi}{a}$ in $\mu_n = \frac{n\pi}{b}$, $m, n \in \mathbb{N}$.
- Časovni del $T'' + \omega_{mn}^2 T = 0$, $\omega_{mn} = \pi c \sqrt{\left(\frac{m}{a}\right)^2 + \left(\frac{n}{b}\right)^2}$.
- Lastne funkcije $u_{mn}(x,y,t) = \sin(\lambda_m x)\sin(\mu_n y)\left(A_{mn}\cos(\omega_{mn} t) + B_{mn}\sin(\omega_{mn} t)\right)$.

Dirichletov robni problem: $\Delta u = 0$ za polpas, $0 < x < 1$, $0 < y < \infty$.

Robni pogoji $u(0,y) = 0$, $u(1,y) = 0$, $u(x,0) = 1$ $\lim_{y\to\infty} u(x,y) = 0$.

- Enačba $u_{xx} + u_{yy} = 0$.
- Ločimo spremenljivke $u(x,y) = X(x)Y(y)$.
- Enačba $\dfrac{X''}{X} + \dfrac{Y''}{Y} = 0$, robni pogoji $X(0) = X(1) = 0$, $|Y(y)|$ je omejena funkcija $0 < y < \infty$.
- $X'' + \lambda_n^2 X = 0$, $\lambda_n = n\pi$ $n \in \mathbb{N}$.
- $X_n(x) = \sin(n\pi x)$ in $Y_n(y) = A_n e^{-n\pi y} + B_n e^{n\pi y}$.
- Koeficienti $B_n = 0$, ker je funkcija $Y(y)$ omejena.
- $u(x,y) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) e^{-n\pi y}$.
- $u(x,0) = \sum_{n=1}^{\infty} A_n \sin(n\pi x) = 1$, $A_n = 2\frac{1-(-1)^2}{n\pi}$.

Graf k prejšnji nalogi $z = \sum_{n=1}^{10} u_n(x,y)$



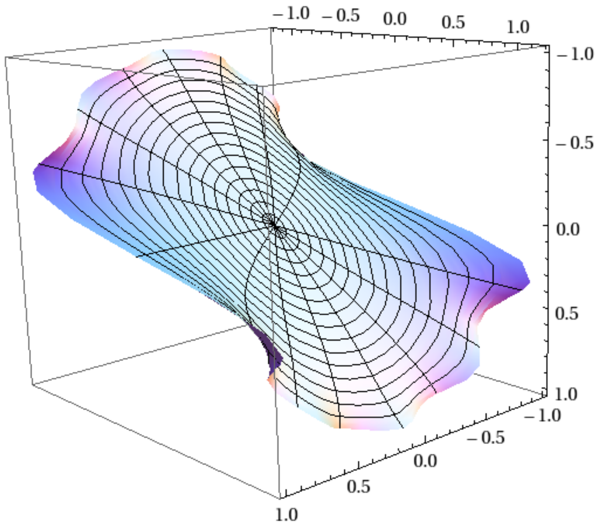
Dirichletov robni problem, $\Delta u = 0$ za krog, $0 \le r \le 1$.

Robni pogoji: $|u(r,\phi)| < \infty$, $u(1,\phi) = f(\phi)$.

- Enačba: $u_{rr} + \dfrac{1}{r}u_r + \dfrac{1}{r^2}u_{\phi\phi} = 0$.
- Ločimo spremenljivke: $u(r,\phi) = R(r)\Phi(\phi)$.
- Enačba: $\dfrac{R''}{R} + \dfrac{1}{r}\dfrac{R'}{R} + \dfrac{1}{r^2}\dfrac{\Phi''}{\Phi} = 0$.
- Robni pogoji: $|R(r)|$ je omejena funkcija, $\Phi(\phi)$ je periodična funkcija s periodo 2π.
- Enačbi:
  - $\Phi'' + \lambda_n \Phi = 0$, $\lambda_n = n^2$, $n \in \mathbb{N}$ in
  - $r^2 R''(r) + rR'(r) - n^2 R(r) = 0$.
- $\Phi_n(\phi) = A_n \sin(n\phi) + B_n \cos(n\phi)$ in $R_n(r) = r^n$.
- $u(r,\phi) = A_0 + \sum_{n=1}^{\infty}\left(A_n\cos(n\phi) + B_n\sin(n\phi)\right) r^n$.
- $u(1,\phi) = f(\phi) = A_0 + \sum_{n=1}^{\infty}\left(A_n\cos(n\phi) + B_n\sin(n\phi)\right)$.

Graf k prejšnji nalogi

$z = \sum_{n=1}^{10} u_n(x,y), \quad f(\phi) = \mathrm{sign}\,(\phi), \quad -\pi \le \phi \le \pi.$



- Enačbe:
  - $\Phi'' + m^2\Phi = 0, \quad m \in \mathbb{N},$
  - $T'' + c^2\lambda^2 T = 0,$
  - $r^2R''(r) + rR'(r) + (\lambda^2 r^2 - m^2)R(r) = 0,$
  - Vpeljemo novo spremenljivko $\xi = \lambda r$ dobimo enačbo
  - $\xi^2 R''(\xi) + \xi R'(\xi) + (\xi^2 - m^2)R(\xi) = 0.$
    Rešitev je $R(\xi) = AJ_m(\xi) + BN_m(\xi)$.
  - $\lambda_{mn} = \xi_{mn}/a,\ \omega_{mn} = c\lambda_{mn}$
    ξ_{mn} je n-ta ničla Besslove funkcije $J_m(\xi)$.
- Rešitve:
  - $\Phi_m(\phi) = \sin(m\phi),$
  - $R_{mn}(r) = J_m(\lambda_{mn} r)$
  - $T_{mn} = A_{mn}\cos(\omega_{mn}t) + B_{mn}\sin(\omega_{mn}t).$
- $u(r,\phi,t) =$
  $\sum_{m,n} \sin(m\phi) J_m(\lambda_{mn} r)\left(A_{mn}\cos(\omega_{mn}t) + B_{mn}\sin(\omega_{mn}t)\right).$

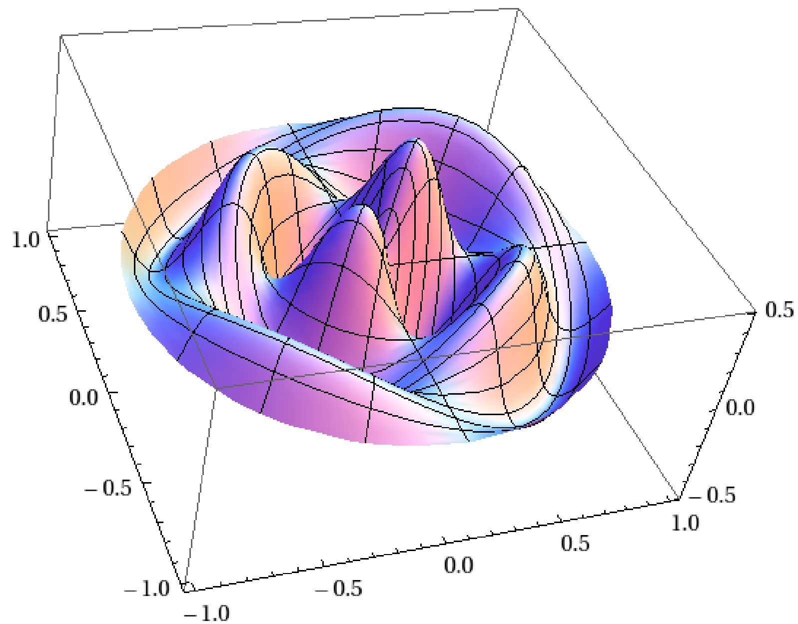
Splošna nihanja vpete okrogle membrane:

$\Delta u = \frac{1}{c^2} u_{tt} \ 0 \le r \le a.$

Robni pogoji: $|u(0,\phi,t)| < \infty$, $u(a,\phi,t) = 0$,
$u(r,\phi + 2\pi, t) = u(r,\phi,t)$, periodičnost.
Začetni pogoji: $u(r,\phi,0) = f(r,\phi)$ in $u_t(r,\phi,0) = g(r,\phi)$

- Enačba: $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\phi\phi} = \frac{1}{c^2}u_{tt}.$
- Ločimo spremenljivke: $u(r,\phi,t) = R(r)\Phi(\phi)T(t)$.
- Enačba: $\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Phi''}{\Phi} = \frac{1}{c^2}\frac{T''}{T}.$
- Robni pogoji: $|R(0)| < \infty$, $R(a) = 0$ in
  $\Phi(\phi)$ je periodična funkcija s periodo 2π.

Graf lastne funkcije $u_{2,3}(r,\phi,0)$.



Prevajanje toplote po palici z izoliranima krajiščema

$$u_{xx} = \frac{1}{\sigma^2}u_t,\ u_x(0,t) = u_x(a,t) = 0,\ u(x,0) = f(x).$$

- $u_{xx}(x,t) = \dfrac{1}{\sigma^2}u_t(x,t)$. $u(x,t) = X(x)T(t)$.
- $\dfrac{X''}{X} = \dfrac{1}{\sigma^2}\dfrac{T'}{T}$.
- Krajevni del: $\frac{X''}{X} = -\lambda^2$
  - Enačba: $X'' + \lambda^2 X = 0$.
  - $X(x) = A\cos(\lambda x) + B\sin(\lambda x)$.
  - Robni pogoji: $X'(x) = -A\lambda\sin(\lambda x) + B\lambda\cos(\lambda x)$.
  - $X(0) = 0 \to B = 0$, $X(a) = 0 \to \lambda a = n\pi$, $n \in \mathbb{N}$.
  - Lastne funkcije $X_n(x) = \cos(\frac{n\pi}{a}x)$.
- Časovni del: $\frac{1}{\sigma^2}\frac{T'}{T} = -\left(\frac{n\pi}{a}\right)^2$. $\mu_n = \sigma\frac{n\pi}{a}$.
  - Enačba $T' + \mu_n^2 T = 0$.
  - $T_n(t) = A_n e^{-\mu_n^2 t}$.
- Lastne funkcije: $u_n(x,t) = A_n\cos(\frac{n\pi}{a}x)e^{-\mu_n^2 t}$.
- Začetni pogoj: $f(x) = \sum_{n=0}^{\infty} A_n\cos(\frac{n\pi}{a}x)$.
- $A_n = \dfrac{2}{a}\displaystyle\int_0^a f(x)\cos(\frac{n\pi}{a}x)dx$, $n > 0$.
- Rešitev: $u(x,t) = A_0 + \displaystyle\sum_{n=1}^{\infty} A_n\cos(\frac{n\pi}{a}x)e^{-\mu_n^2 t}$.
- Stacionarna rešitev $\displaystyle\lim_{t\to\infty} u(x,t) = A_0 = \frac{1}{a}\int_0^a f(x)dx$.

Temperatura na krajiščih je konstantna.

$$u_{xx} = \frac{1}{\sigma^2}u_t,\ u(0,t) = \tau_0,\ u(a,t) = \tau_a \text{ in } u(x,0) = f(x).$$

- Piščimo staconarno rešitev, $\overline{u}_t = 0$, to je rešitev neodvisna od časa, $\overline{u}_{xx} = 0$.
- Rešitev je linearna funkcija: $\overline{u}(x) = \tau_0 + \frac{\tau_a - \tau_0}{a}x$.
- $u(x,t) = \overline{u}(x) + v(x,t)$, $v(0,t) = v(a,t) = 0$.
- $v(x,t) = X(x)T(t)$. $\frac{X''}{X} = \frac{1}{\sigma^2}\frac{T'}{T} = -\lambda^2$.
- $X(x) = A\cos(\lambda x) + B\sin(\lambda x)$, $X(0) = X(a) = 0$.
- $v_n(x,t) = A_n\sin(\frac{n\pi}{a}x)e^{-(\sigma\frac{n\pi}{a})^2 t}$.
- Rešitev: $u(x,t) = \tau_0 + \dfrac{\tau_a - \tau_0}{a}x + \displaystyle\sum_{n=1}^{\infty} A_n\sin(\frac{n\pi}{a}x)e^{-(\sigma\frac{n\pi}{a})^2 t}$.
- $A_n = \dfrac{2}{a}\displaystyle\int_0^a \left(f(x) - \left(\tau_0 + \frac{\tau_a - \tau_0}{a}x\right)\right)\sin(\frac{n\pi}{a}x)dx$.

Reši Poissonov robni problem $\Delta u = f$ v prostoru. Iščemo rešitev, ki je odvisna le od $r = \sqrt{x^2 + y^2 + z^2}$

$$\Delta u = f, \quad u(1) = 1, \quad u(0) < \infty, \quad f(r) = \begin{cases} 1 & 0 \le r \le 1 \\ 0 & \text{drugod} \end{cases}$$

- Laplaceov operator $\Delta u(r) = \dfrac{\partial^2 u}{\partial u^2} + \dfrac{2}{r}\dfrac{\partial u}{\partial r^2}$.
- $r^2 u''(r) + 2ru'(r) = \begin{cases} r^2 & 0 \le r < 1 \\ 0 & r \le 1 < \infty \end{cases}$, $u(1) = 1$, $u(0) < \infty$.
- Eulerjeva diferencialna enačba, $r = e^x$.
- $u''(x) + u'(x) = \begin{cases} e^{2x} & -\infty < x \le 0 \\ 0 & 0 < x < \infty \end{cases}$, $u(0) = 1$, $\displaystyle\lim_{x\to-\infty} u(x) < \infty$.

- $u(r) = \dfrac{-20c_1 + r^5\theta(r) - \left(r^5 - 5r + 4\right)\theta(r-1)}{20r} + c_2$
- $u(r) = \dfrac{1}{20} r^4 \theta(r) - \dfrac{1}{20r}\left(r^5 - 5r + 4\right)\theta(r-1)$

Reši Poissonov robni problem $\Delta u = f$ v prostoru. Iščemo rešitev, ki je odvisna le od $r = \sqrt{x^2 + y^2 + z^2}$

$$\Delta u = 1, \quad 1 < r < 2, \quad u(1) = 0, \quad u(2) = 0$$

- Laplaceov operator $\Delta u(r) = \dfrac{\partial^2 u}{\partial u^2} + \dfrac{2}{r}\dfrac{\partial u}{\partial r^2}$.
- $r^2 u''(r) + 2ru'(r) = r^2$, $u(1) = 0$, $u(2) = 0$.
- Splošna rešitev $u(r) = c_1 + c_2 r^{-1} + \frac{1}{6} r^2$.
- $u(r) = \dfrac{r^3 - 7r + 6}{6r}$.

Reševanje parcialnih diferencialnih enačb s pomočjo Laplaceove transformacije.

Reši diferencialno enačbo

$$\frac{\partial u(x,t)}{\partial x} + x\frac{\partial u(x,t)}{\partial t} = 0, \quad t > 0, \quad u(x,0) = 0, \quad u(0,t) = t^2.$$

- Laplaceova transformacija enačbe:
- $\dfrac{dU(x,s)}{dx} + x\left(sU(x,s) - u(x,0)\right) = 0$.
- $\displaystyle\int \frac{dU}{U} = -\int xs\,dx \to U(x,s) = F(s)e^{-s\frac{x^2}{2}}$.
- $u(x,t) = f(t - \dfrac{x^2}{2})$, $\quad u(0,t) = t^2$, $\quad \to$
- $u(x,t) = (t - \dfrac{x^2}{2})^2\theta(t - \dfrac{x^2}{2})$.

Reševanje parcialnih diferencialnih enačb s pomočjo Laplaceove transformacije.

Reši diferencialno enačbo

$$\frac{\partial u(x,t)}{\partial x} + 2x\frac{\partial u(x,t)}{\partial t} = 2x, \quad t > 0, \quad u(x,0) = 1, \quad u(0,t) = 1.$$

- Laplaceova transformacija enačbe:
- $\dfrac{dU(x,s)}{dx} + 2x\left(sU(x,s) - u(x,0)\right) = 2\dfrac{x}{s}$.
- $\dfrac{dU(x,s)}{dx} + 2xsU(x,s) = \dfrac{2x}{s} + 2x \to$
- $U(x,s) = F(s)e^{-sx^2} + \dfrac{1+s}{s^2}$.
- $u(x,t) = f(t - x^2) + 1 + t$
  $u(x,0) = f(t) + 1 + t = 1, \to f(t) = -t$.
- $u(x,t) = -(t - x^2)\theta(t - x^2) + (1+t)\theta(t)$.