Izpit Matematika IV

29.avgust 2006

Rešitve

1. naloga

$$(x+1)\sum_{n=1}^{\infty} C_n n x^{n-1} = 3\sum_{n=0}^{\infty} C_n x^n$$

$$\sum_{n=1}^{\infty} C_n n x^n + \sum_{n=1}^{\infty} C_n n x^{n-1} - 3\sum_{n=0}^{\infty} C_n x^n = 0$$

$$\sum_{n=1}^{\infty} C_n n x^n + \sum_{n=0}^{\infty} C_{n+1}(n+1) x^n - 3\sum_{n=0}^{\infty} C_n x^n = 0$$

$$(C_1 - 3C_0) + \sum_{n=1}^{\infty} \Big[C_{n+1}(n+1) + C_n n - 3C_n\Big] x^n = 0$$

$C_1 - 3C_0 = 0 \quad , \quad C_{n+1} = \frac{3-n}{n+1} C_n \ , \ n = 1, 2, \ldots$

C_0 poljuben

$C_1 = 3C_0$

$C_2 = \frac{2}{2} C_1 = 3C_0$

$C_3 = \frac{1}{3} C_2 = C_0$

$C_4 = 0 \quad , \quad C_5 = C_6 = \cdots = 0$

$$\boxed{y = C_0\big(1 + 3x + 3x^2 + x^3\big)}$$

2. naloga

Kot običajno, označimo *Laplacovi* transformiranki neznanih funkcij:
$\mathcal{L}[x(t)] = X(s)\ ,\ \mathcal{L}[y(t)] = Y(s)$

$sX - 2 + 3X - 4Y = \frac{9}{s-2} \quad / \cdot (-2)$

$2X + sY - 0 - 3Y = \frac{3}{s-2} \quad / \cdot (s+3)$

$4 + [8 + (s-3)(s+3)]Y = \frac{-18}{s-2} + \frac{3(s+3)}{s-2}$

$[s^2 - 9 + 8]Y = \frac{3s+9-18}{s-2} - 4$

$Y = \frac{1}{s^2-1} \cdot \frac{3s-9-4s+8}{s-2} = \frac{-s-1}{(s-1)(s+1)(s-2)} = \frac{-1}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}$

$A(s-2) + B(s-1) = -1$

$s = 1 \quad \rightarrow \quad A = 1$

$s = 2 \quad \rightarrow \quad B = -1$

$Y = \frac{1}{s-1} - \frac{1}{s-2}$

$$\boxed{y(t) = e^t - e^{2t}}$$

Iz druge diferencialne enačbe:

$x = \frac{1}{2}[3e^{2t} + 3y - y'] = \frac{1}{2}[3e^{2t} + 3e^t - 3e^{2t} - e^t + 2e^{2t}]$

$$\boxed{x(t) = e^t + e^{2t}}$$

3. naloga

$$u = F(x)G(y)$$
$$F''G + FG'' = 0$$
$$\frac{F''}{F} = -\frac{G''}{G} = \lambda^2$$

$$G'' + \lambda^2 G = 0$$
$$G(y) = A\cos(\lambda y) + B\sin(\lambda y)$$
$$G'(y) = -A\lambda\sin(\lambda y) + B\lambda\cos(\lambda y)$$
$$G'(0) = 0 \quad \rightarrow \quad B = 0$$
$$G'(b) = 0 \quad \rightarrow \quad \lambda_n = \tfrac{n\pi}{b}$$

$$G_n(y) = A_n\cos(\lambda_n y)\ ,\ n = 1, 2, \ldots$$
$$G_0(y) = A_0$$

$$F'' - \lambda_n^2 F = 0$$

$$F_n(x) = C_n\,\mathrm{sh}(\lambda_n x) + D_n\,\mathrm{ch}(\lambda_n x)$$
$$F_0(x) = C_0 x + D_0$$

$$u(x, y) = (c_0 x + d_0) + \sum_{n=1}^{\infty}\Big[c_n\,\mathrm{sh}(\lambda_n x) + d_n\,\mathrm{ch}(\lambda_n x)\Big]\cos(\lambda_n y)$$

$x = 0 \quad \rightarrow \quad d_0 + \sum_{n=1}^{\infty} d_n \cos\tfrac{n\pi y}{b}$ je *Fourier kosinusna* vrsta za konstanto 0.

Zato $d_0 = d_n = 0$

$x = a \quad \rightarrow \quad c_0 a + \sum_{n=1}^{\infty} c_n\,\mathrm{sh}\tfrac{n\pi a}{b}\cos\tfrac{n\pi y}{b}$ je *Fourier kosinusna* vrsta za konstanto V_0.

Zato $c_n = 0$ in $c_0 a = V_0$.

$$\boxed{u(x, y) = \frac{V_0}{a}x}$$

4. naloga

Zapišemo *Eulerjevo* diferencialno enačbo za funkcional

$$F(y) = \int_0^1 \left(y'^2 + x^2 + \lambda y^2\right) dx$$

$2\lambda y - (2y')' = 0$
$y'' - \lambda y = 0$

Karakterisična enačba $r^2 - \lambda = 0$ ima korena $r_{1,2} = \pm\sqrt{\lambda}$
Ločimo tri primere:

$\lambda = k^2 > 0$
$y = Ae^{kx} + Be^{-kx}$
$x = 0 \quad \to \quad A + B = 0 \quad \to \quad B = -A$
$x = 1 \quad \to \quad A\left(e^k - e^{-k}\right) = 0 \quad \to \quad A = B = 0 \quad \to \quad y = 0$
Ni ekstremala, ker ne zadošča pogoju $\int_0^1 y^2 = 2$

$\lambda = 0$
$y = Ax + B$
$x = 0 \quad \to \quad B = 0$
$x = 1 \quad \to \quad A = 0 \quad \to \quad y = 0$
Ni ekstremala, ker ne zadošča pogoju $\int_0^1 y^2 = 2$

$\lambda = -k^2 < 0$
$y = A\cos kx + B\sin kx$

$x = 0 \quad \to \quad A = 0$
$x = 1 \quad \to \quad k = n\pi \quad \to \quad y = B\sin(n\pi x)\ ,\ n \in N$
y vstavimo v pogoj $\int_0^1 y^2 = 2$ in določimo konstanto B :

$$\int_0^1 B^2 \sin^2(n\pi x)\, dx = B^2 \frac{1}{2} = 2 \quad \to \quad B = \pm 2$$

$$\boxed{y = \pm 2\sin(n\pi x)\ ,\ n \in N}$$

5. naloga

$$P = 1 - P(\text{ vsi so dobri }) = 1 - 0,9^{12} = \boxed{0,7176}$$