Izpit Matematika IV

22. junij 2007

Rešitve

1. naloga

$\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = t$

$\mathcal{L}^{-1}\left[\frac{e^{-s}}{s^2}\right] = (t-1)u_1(t)$

$\mathcal{L}^{-1}\left[\frac{1}{s^2-1}\right] = \mathcal{L}^{-1}\left[\frac{1}{2}(\frac{1}{s-1} - \frac{1}{s+1})\right] = \frac{1}{2}(e^t - e^{-t}) = \sinh t$

$\mathcal{L}^{-1}\left[\frac{e^{-2s}}{s^2-1}\right] = \sinh(t-2)u_2(t)$

$$\boxed{f(t) = \cos(2t) - (t-1)u_1(t) - \sinh(t-2)u_2(t)}$$

2. naloga

$$y = \sum_{n=0}^{\infty} C_n x^n$$

$$\sum_{n=1}^{\infty} C_n n x^{n-1} = 2x \sum_{n=0}^{\infty} C_n x^n$$

$$\sum_{n=1}^{\infty} C_n n x^{n-1} = 2 \sum_{n=0}^{\infty} C_n x^{n+1}$$

$$\sum_{n=0}^{\infty} C_{n+1}(n+1) x^n = 2 \sum_{n=1}^{\infty} C_{n-1} x^n$$

$C_1 = 0 \quad , \quad C_{n+1} = \frac{2}{n+1} C_{n-1} \ , \ n = 1, 2, \ldots$

Vsi lihi so enaki 0: $C_{2n+1} = 0$
Sodi se dajo izraziti s C_0 :

$C_2 = C_0$

$C_4 = \frac{2}{4} C_2 = \frac{1}{2!} C_0$

$C_6 = \frac{2}{6} C_4 = \frac{1}{3!} C_0$

$C_8 = \frac{2}{8} C_6 = \frac{1}{4!} C_0$

$\ldots$

$C_{2n} = \frac{1}{n!} C_0$

$$\boxed{y = C_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{n!}}$$

3. naloga

$u(x,t) = F(x)G(t)$
$F(x)G'(t) = F''(x)G(t)$
$\frac{G'(t)}{G(t)} = \frac{F''(x)}{F(x)} = -\lambda^2$

Diferencialna enačba za $F(x)$ in robni pogoji:

$F''(x) + \lambda^2 F(x) = 0$
$F(x) = A\cos\lambda x + B\sin\lambda x$
$x = 0 \quad \to \quad F(0) = 0 \quad \to \quad A = 0$
$x = 1 \quad \to \quad F(1) = 0 \quad \to \quad \sin\lambda = 0 \quad \to \quad \lambda_n = n\pi\ ,\ n = 1, 2, \ldots$

$F_n(x) = B_n \sin(n\pi x)$

Diferencialna enačba za $G(t)$:

$\frac{G'}{G} = -\lambda_n^2$
$G_n(t) = C_n e^{-\lambda_n^2 t}$

Vsota po n in določitev koeficientov iz začetnega pogoja:

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-(n\pi)^2 t} \sin(n\pi x)$$

$$t = 0 \quad \to \quad \sum_{n=1}^{\infty} b_n \sin(n\pi x) = 1$$

Funkcijo $f(x) = 1$ razvijemo v *Fourierovo sinusno* vrsto na intervalu $(0,1)$.
Razvoj prepišemo iz priročnika, glej *Bronstein* stran 853, primer 5:

$1 = \frac{4}{\pi}\Big(\sin(\pi x) + \frac{\sin(3\pi x)}{3} + \frac{\sin(5\pi x)}{5} + \cdots\Big)$

Od tu razberemo, da je $b_n = 0\ ,\ n = 2k \quad , \quad b_n = \frac{4}{n\pi}\ ,\ n = 2k+1$

$$\boxed{u(x,t) = \frac{4}{\pi} \sum_{n=1,3,5\ldots}^{\infty} \frac{\sin(n\pi x)}{n} e^{-(n\pi)^2 t}}$$

4. naloga

$$2y - 2\sin x - (2y')' = 0$$
$$y'' - y = -\sin x$$

$$r^2 - 1 = 0$$
$$r_{1,2} = \pm 1$$
$$y_h = A\cosh x + B\sinh x$$

$$y_p = C\sin x$$
$$y_p'' = -C\sin x$$
$$-C\sin x - C\sin x = -\sin x$$
$$C = \tfrac{1}{2}$$

$$\boxed{y = A\cosh x + B\sinh x + \tfrac{1}{2}\sin x}$$

5. naloga

$$A\int_0^1 x(1-x)\,dx = A\Big[\frac{x^2}{2} - \frac{x^3}{3}\Big]\Big|_0^1 = \tfrac{A}{6} = 1 \quad \to \quad \boxed{A = 6}$$

$$P(X \le \tfrac{1}{2} \mid \tfrac{1}{3} \le X \le \tfrac{2}{3}) = P(X \le \tfrac{2}{3}) = \int_0^{\frac{2}{3}} 6x(1-x)\,dx = 6\Big[\frac{x^2}{2} - \frac{x^3}{3}\Big]\Big|_0^{\frac{2}{3}} = \boxed{\frac{20}{27}}$$