Izpit Matematika IV

18.september 2009

Rešitve

1. naloga

$f(t) = \sin(\omega t + \theta) = \sin\omega t\cos\theta + \cos\omega t \sin\theta$

$F(s) = \boxed{\frac{1}{s2+\omega^2}(\omega\cos\theta + s\sin\theta)}$

2. naloga

$$y = \sum_{n=0}^{\infty} C_n x^n$$

$$y' = \sum_{n=1}^{\infty} C_n n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} C_n n(n-1) x^{n-2}$$

$$\sum_{n=2}^{\infty} C_n n(n-1) x^{n-1} + \sum_{n=0}^{\infty} C_n x^n = 0$$

$$\sum_{n=1}^{\infty} C_{n+1}(n+1)n x^n + \sum_{n=0}^{\infty} C_n x^n = 0$$

$$C_0 + \sum_{n=1}^{\infty} \Big[C_{n+1}(n+1)n + C_n\Big] x^n = 0$$

$y(0) = 0 \quad \rightarrow \quad C_0 = 0$
$y'(0) = 1 \quad \rightarrow \quad C_1 = 1$
$C_{n+1} = -\frac{C_n}{(n+1)n} \quad , \; n = 1, 2, 3, \ldots$

Če opazujemo nekaj prvih koeficientov, se da odkriti splošno formulo za C_n :

$C_2 = -\frac{1}{2}$

$$C_3 = \frac{1}{2^2 3}$$

$$C_4 = -\frac{1}{2^2 3^2 4}$$

$$C_5 = \frac{1}{2^2 3^2 4^2 5}$$

$$C_n = (-1)^{n+1} \frac{1}{[(n-1)!]^2 n}$$

$$\boxed{y = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{[(n-1)!]^2 n} x^n}$$

3. naloga

V nalogi je predpisan nehomogeni robni pogoj $\frac{\partial u(\pi,t)}{\partial x} = A$, zato vpeljemo novo neznano funkcijo z enačbo $u(x,t) = v(x,t) + w(x)$, kjer bo nova neznanka funkcija $v(x,t)$. Od funkcije $w(x)$ zahtevamo, da prevzame nehomogen robni pogoj, dobra je kar premica $w = Ax$. Zatorej vpeljemo novo neznanko $u(x,t) = v(x,t) + Ax$ in prepišemo celoten problem za neznanko $v(x,t)$:

$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}$ pri pogojih $v(0,t) = 0$, $\frac{\partial v(\pi,t)}{\partial x} = 0$, $v(x,0) = -Ax$

Separacija spremenljivk $v(x,t) = F(x)G(t)$
$F(x)G'(t) = F''(x)G(t)$
$\frac{G'(t)}{G(t)} = \frac{F''(x)}{F(x)} = -\lambda^2$

Diferencialna enačba za $F(x)$ in robni pogoji:
$F''(x) + \lambda^2 F(x) = 0$
$F(x) = A\cos\lambda x + B\sin\lambda x$
$x = 0 \quad \to \quad F(0) = 0 \quad \to \quad A = 0$
$x = \pi \quad \to \quad F'(\pi) = 0 \quad \to \quad \cos\lambda\pi = 0 \quad \to \quad \lambda_n = n + \frac{1}{2}$, $n = 0, 1, 2, \ldots$
$F_n(x) = B_n \sin(n + \frac{1}{2})x$

Diferencialna enačba za $G(t)$:
$\frac{G'}{G} = -\lambda_n^2$
$G_n(t) = C_n e^{-\lambda_n^2 t}$

Vsota po n in določitev koeficientov iz začetnega pogoja:

$$v(x,t) = \sum_{n=0}^{\infty} D_n e^{-(n+\frac{1}{2})^2 t} \sin(n + \tfrac{1}{2})x$$

$$t = 0 \quad \to \quad \sum_{n=0}^{\infty} D_n \sin(n + \tfrac{1}{2})x = -Ax$$

D_n so koeficienti v razvoju funkcije $-Ax$ v *Fourierovo* vrsto po funkcijah $\sin(n + \frac{1}{2})x$:

$$D_n = \frac{2}{\pi} \int_0^{\pi} (-Ax) \sin(n + \frac{1}{2})x \, dx = \frac{8A}{\pi} \frac{(-1)^{n+1}}{(1+2n)^2}$$

Vstavimo D_n v vrsto za $v(x,t)$ in zapišemo rezultat za originalno neznanko $u(x,t)$:

$$\boxed{u(x,t) = Ax + \frac{8A}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(1+2n)^2} e^{-(n+\frac{1}{2})^2 t} \sin(n + \tfrac{1}{2})x}$$

4. naloga

$2y - 2\sin x - (2y')' = 0$

$y'' - y = -\sin x$

Karakteristična enačba $\lambda^2 - 1 = 0 \quad \rightarrow \quad \lambda_{1,2} = \pm 1$

Nastavek za partikularno rešitev $y = C\sin x \quad \rightarrow \quad C = \frac{1}{2}$

$$\boxed{y = Ae^x + Be^{-x} + \tfrac{1}{2}\sin x}$$

5. naloga

$$A\int_0^1 x(1-x)\,dx = A\Big[\frac{x^2}{2} - \frac{x^3}{3}\Big]\Big|_0^1 = \tfrac{A}{6} = 1 \quad \rightarrow \quad \boxed{A = 6}$$

$$P(X \le \tfrac{1}{2} \mid \tfrac{1}{3} \le X \le \tfrac{2}{3}) = P(X \le \tfrac{2}{3}) = \int_0^{\frac{2}{3}} 6x(1-x)\,dx = 6\Big[\frac{x^2}{2} - \frac{x^3}{3}\Big]\Big|_0^{\frac{2}{3}} = \boxed{\frac{20}{27}}$$