Izpit Matematika IV

28.junij 2010

Rešitve

1. naloga

$$\begin{aligned} s^2Y_1 - 2s - 3 &= Y_1 + 3Y_2 \\ s^2Y_2 - s - 2 &= 4Y_1 - \tfrac{4}{s-1} \end{aligned}$$

Iz prve enačbe izrazimo Y_2 in vstavimo v drugo enačbo.

$Y_2 = \frac{1}{3}\big(s^2Y_1 - Y_1 - 2s - 3\big)$

$\frac{s^4}{3}Y_1 - \frac{s^2}{3}Y_1 - \frac{2}{3}s^3 - s^2 - s - 2 = 4Y_1 - \frac{4}{s-1}$

$\Big(\frac{s^4}{3} - \frac{s^2}{3} - 4\Big)Y_1 = \frac{2}{3}s^3 + s^2 + s + 2 - \frac{4}{s-1}$

$Y_1 = \frac{2s^3+3s^2+3s+6}{s^4-s^2-12} - \frac{12}{(s-1)(s^4-s^2-12)} = \frac{2s^4+s^3+3s-18}{(s-1)(s-2)(s+2)(s^2+3} = \frac{2s^3-3s^2+6s-9}{(s-1)(s-2)(s+3)}$

$Y_1 = \frac{A}{s-1} + \frac{B}{s-2} + \frac{Cs+D}{s^2+3}$

$A(s-2)(s^2+3) + V(s-1)(s^2+3) + (Cs+D)(s-1)(s-2) = 2s^3 - 3s^2 + 6s - 9$

$s = 2 \quad \to \quad 7B = 7 \quad \to \quad B = 1$
$s = 1 \quad \to \quad -4A = -4 \quad \to \quad A = 1$
$ks^3 \quad \to \quad 1 + 1 + C = 2 \quad \to \quad C = 0$
$ks^2 \quad \to \quad -2 - 1 + D = -3 \quad \to \quad D = 0$

$Y_1 = \frac{1}{s-1} + \frac{1}{s-2}$

$\boxed{y_1 = e^t + e^{2t}}$

$y_2 = \frac{1}{3}(y_1'' - y_1) = \frac{1}{3}(e^t + 4e^{2t} - e^t - e^{2t})$

$\boxed{y_2 = e^{2t}}$

2. naloga

$$y = \sum_{n=0}^{\infty} C_n x^n$$
$$y' = \sum_{n=1}^{\infty} C_n n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} C_n n(n-1) x^{n-2}$$

$$\sum_{n=2}^{\infty} C_n n(n-1) x^{n-1} + 2\sum_{n=1}^{\infty} C_n n x^{n-1} + \sum_{n=0}^{\infty} C_n x^{n+1}$$
$$\sum_{n=2}^{\infty} C_n n(n-1) x^{n-1} + 2\sum_{n=1}^{\infty} C_n n x^{n-1} + \sum_{n=2}^{\infty} C_{n-2} x^{n-1}$$
$$2C_1 + \sum_{n=2}^{\infty} \Big[C_n n(n-1) + 2C_n n + C_{n-2}\Big] x^{n-1} = 0$$

$C_0 =$ poljuben , $C_1 = 0$ in rekurzijska formula za koeficiente:

$$C_n = \frac{-1}{n(n+1)} C_{n-2} \quad , \quad n = 2, 3, \ldots$$

Vsi lihi koeficienti so enaki 0, sodi so proporcionalni koeficientu C_0.
Ker rabimo samo eno rešitev, izberimo $C_0 = 1$. Nekaj naslednjih koeficientov:

$C_2 = -\frac{1}{2\cdot 3}$, $C_4 = \frac{1}{2\cdot 3\cdot 4\cdot 5}$, $C_6 = -\frac{1}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}$

Spreminjanje koeficientov se da dešifrirati, zapisemo le sode člene:

$C_{2n} = (-1)^n \frac{1}{(2n+1)!}$

$$\boxed{y = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n}}$$

To je skoraj vrsta za funkcijo sinus:

$$y = \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \boxed{\frac{\sin x}{x}}$$

3. naloga

Označimo $\mathcal{L}\big[u(x,t)\big] = U(x,s)$
Problem za neznanko $u(x,t)$ se transfomira v problem za neznanko $U(x,s)$:

$$x\frac{\partial U}{\partial x} + sU - 0 = \frac{x}{s^2} \quad , \quad U(0,s) = 0$$

To rešimo kot navadno dif. enačbo in na s gledamo kot konstanto.
Je linearna dif.enačba 1.reda; najprej homogena in nato variacija konstante.

$$x\frac{dU}{dx} = -sU$$
$$\int \frac{dU}{U} = \int -\frac{s}{x}dx$$
$$\ln U = -s\ln x + \ln C$$
$$U_h = \frac{C}{x^s}$$

$$U_p = \frac{C(x)}{x^s}$$
$$x\frac{C'}{x^s} + x\frac{C(-s)}{x^{s+1}} + s\frac{C}{x^s} = \frac{x}{s^2}$$
$$C' = \frac{x^s}{s^2}$$
$$C = \frac{x^{s+1}}{(s+1)s^2}$$
$$U_p = \frac{x}{(s+1)s^2}$$
$$U = U_h + U_p = \frac{C}{x^s} + \frac{x}{(s+1)s^2}$$
$$x = 0 \quad \rightarrow \quad C = 0$$
$$u(x,t) = \mathcal{L}^{-1}\Big[U(x,s)\Big] = \mathcal{L}^{-1}\Big[\frac{x}{(s+1)s^2}\Big]$$

$$\operatorname*{Res}_{s=-1} = \lim_{s\to -1}\frac{xe^{st}}{s^2} = xe^{-t}$$
$$\operatorname*{Res}_{s=0} = \lim_{s\to 0}\Big[\frac{xe^{st}}{s+1}\Big]' = \lim_{s\to 0}\frac{xe^{st}t(s+1) - xe^{st}}{(s+1)^2} = xt - x$$

$$\boxed{u(x,t) = x(t - 1 + e^{-t})}$$

4. naloga

$2y' - 32y - (2y' + 2y)' = 0$
$2y' - 32y - 2y'' - 2y' = 0$
$y'' + 16y = 0$

$r^2 + 16 = 0$
$r_{1,2} = \pm 4i$

$$\boxed{y = A\cos(4x) + B\sin(4x)}$$

5. naloga

$(\text{ vsota deljiva s } 4) = (\text{vsota=4}) \cup (\text{vsota=8}) \cup (\text{vsota=12}) =$
$(13 \cup 22 \cup 31) \; \cup \; (26 \cup 35 \cup 44 \cup 53 \cup 62) \; \cup \; (66)$

$P(\text{ vsota deljiva s } 4\,) = \frac{3+5+1}{36} = \boxed{\frac{1}{4}}$