Izpit Matematika IV

12.september 2011

Rešitve

1. naloga

$f(t) = \sin(\omega t + \theta) = \sin \omega t \cos \theta + \cos \omega t \sin \theta$

$F(s) = \boxed{\frac{1}{s2 + \omega^2} \big(\omega \cos \theta + s \sin \theta\big)}$

2. naloga

$$y = \sum_{n=0}^{\infty} C_n x^n$$
$$y' = \sum_{n=1}^{\infty} C_n n x^{n-1}$$
$$y'' = \sum_{n=2}^{\infty} C_n n(n-1) x^{n-2}$$

$$\sum_{n=2}^{\infty} C_n n(n-1) x^{n-2} + \sum_{n=0}^{\infty} C_n x^{n+1} = 0$$
$$\sum_{n=0}^{\infty} C_{n+2}(n+2)(n+1) x^{n} + \sum_{n=1}^{\infty} C_{n-1} x^{n} = 0$$
$$2C_2 + \sum_{n=1}^{\infty} \Big[C_{n+2}(n+2)(n+1) + C_{n-1}\Big] x^n = 0$$

$$C_2 = 0 \quad , \quad C_{n+2} = \frac{-1}{(n+2)(n+1)} C_{n-1}$$

$$y(0) = 0 \quad \rightarrow \quad C_0 = C_3 = C_6 \cdots = 0$$
$$y'(0) = 1 \quad \rightarrow \quad C_1 = 1 \quad , \quad C_4 = -\tfrac{1}{12} \quad , \quad C_7 = \tfrac{1}{504}$$
$$C_2 = 0 \quad \rightarrow \quad C_5 = C_8 = C_{11} \cdots = 0$$

$$\boxed{y = x - \frac{1}{12}x^4 + \frac{1}{504}x^7 - \cdots}$$

3. naloga

V nalogi je predpisan nehomogeni robni pogoj $\frac{\partial u(\pi,t)}{\partial x} = A$, zato vpeljemo novo neznano funkcijo z enačbo $u(x,t) = v(x,t) + w(x)$, kjer bo nova neznanka funkcija $v(x,t)$. Od funkcije $w(x)$ zahtevamo, da prevzame nehomogen robni pogoj, dobra je kar premica $w = Ax$. Zatorej vpeljemo novo neznanko $u(x,t) = v(x,t) + Ax$ in prepišemo celoten problem za neznanko $v(x,t)$:

$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}$ pri pogojih $v(0,t) = 0$, $\frac{\partial v(\pi,t)}{\partial x} = 0$, $v(x,0) = -Ax$

Separacija spremenljivk $v(x,t) = F(x)G(t)$

$F(x)G'(t) = F''(x)G(t)$

$\frac{G'(t)}{G(t)} = \frac{F''(x)}{F(x)} = -\lambda^2$

Diferencialna enačba za $F(x)$ in robni pogoji:

$F''(x) + \lambda^2 F(x) = 0$

$F(x) = A\cos\lambda x + B\sin\lambda x$

$x = 0 \quad \rightarrow \quad F(0) = 0 \quad \rightarrow \quad A = 0$

$x = \pi \quad \rightarrow \quad F'(\pi) = 0 \quad \rightarrow \quad \cos\lambda\pi = 0 \quad \rightarrow \quad \lambda_n = n + \frac{1}{2}$, $n = 0, 1, 2, \ldots$

$F_n(x) = B_n \sin(n + \frac{1}{2})x$

Diferencialna enačba za $G(t)$:

$\frac{G'}{G} = -\lambda_n^2$

$G_n(t) = C_n e^{-\lambda_n^2 t}$

Vsota po n in določitev koeficientov iz začetnega pogoja:

$$v(x,t) = \sum_{n=0}^{\infty} D_n e^{-(n+\frac{1}{2})^2 t} \sin(n + \tfrac{1}{2})x$$

$$t = 0 \quad \rightarrow \quad \sum_{n=0}^{\infty} D_n \sin(n + \tfrac{1}{2})x = -Ax$$

D_n so koeficienti v razvoju funkcije $-Ax$ v *Fourierovo* vrsto po funkcijah $\sin(n + \frac{1}{2})x$. Razvoj lahko prepišemo iz priročnika (*Bronstein* str 853 primer 4) ob zamenjavi skale $x \rightarrow \frac{x}{2}$; ali pa izračunamo integrale:

$$D_n = \tfrac{2}{\pi} \int_0^{\pi} (-Ax) \sin(n + \frac{1}{2})x \, dx = \frac{8A}{\pi} \frac{(-1)^{n+1}}{(1+2n)^2}$$

Vstavimo D_n v vrsto za $v(x,t)$ in zapišemo rezultat za originalno neznanko $u(x,t)$:

$$\boxed{u(x,t) = Ax + \frac{8A}{\pi} \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(1+2n)^2} e^{-(n+\frac{1}{2})^2 t} \sin(n + \frac{1}{2})x}$$

4. naloga

$2y - 2\sin x - (2y')' = 0$

$y'' - y = -\sin x$

$r^2 - 1 = 0$

$r_{1,2} = \pm 1$

$y_h = A\cosh x + B\sinh x$

$y_p = C\sin x$

$y_p'' = -C\sin x$

$-C\sin x - C\sin x = -\sin x$

$C = \frac{1}{2}$

$$\boxed{y = A\cosh x + B\sinh x + \frac{1}{2}\sin x}$$

5. naloga

Označimo dogodke:
$Hi = (N_1 = i)\ ,\ i = 1, 2, ...6$
$A = (N_2 = 3)$
Uporabimo formulo $P(A) = \sum_i P(H_i)P(A/H_i)$

Potrebne verjetnosti zberemo v tabeli:

i	$P(H_i)$	$P(A/H_i)$
1	1/6	0
2	1/6	0
3	1/6	1/3
4	1/6	1/4
5	1/6	1/5
6	1/6	1/6

$$P(A) = \frac{1}{6}\left(\frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}\right) = \boxed{\frac{19}{120}}$$