Izpit Matematika IV

5.julij 2012

Rešitve

1. naloga

$$\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = t$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2-1}\right] = \mathcal{L}^{-1}\left[\frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right)\right] = \frac{1}{2}(e^t - e^{-t}) = \sinh t$$

$$\mathcal{L}^{-1}\left[\frac{s}{s^2+4} - \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2-1}\right] = \boxed{\cos 2t - (t-1)u_1(t) - \sinh(t-2)u_2(t)}$$

2. naloga

$$y = \sum_{n=0}^{\infty} C_n x^n$$
$$y' = \sum_{n=1}^{\infty} C_n n x^{n-1}$$

$$\sum_{n=1}^{\infty} C_n n x^{n-1} - \sum_{n=1}^{\infty} C_n n x^n = 1 + x - \sum_{n=0}^{\infty} C_n x^n$$

$$\sum_{n=0}^{\infty} C_{n+1}(n+1) x^n - \sum_{n=1}^{\infty} C_n n x^n = 1 + x - \sum_{n=0}^{\infty} C_n x^n$$

Vrsti na levi in desni morata imeti enake koeficiente pri vseh potencah x^n. Ker sta na desni strani dva posebna člena, izenačimo koeficienta pri x^0 in x^1 posebej, potem pa še izenačimo koeficienta pri splošni potenci x^n :

$y(0) = 0 \quad \to \quad C_0 = 0$
$kx^0 \quad \to \quad C_1 = 1 - C_0 \quad \to \quad C_1 = 1$
$kx^1 \quad \to \quad C_2 2 - C_1 = 1 - C_1 \quad \to \quad C_2 = \frac{1}{2}$
$n \geq 2 \quad \to \quad (n+1)C_{n+1} - nC_n = -C_n \quad \to \quad C_{n+1} = \frac{n-1}{n+1} C_n$

Opazujemo nekaj prvih koeficientov:

$C_2 = \frac{1}{2}$
$C_3 = \frac{1}{2 \cdot 3}$
$C_4 = \frac{1}{3 \cdot 4}$
$C_5 = \frac{1}{4 \cdot 5}$

Sklepamo, da velje formula

$C_n = \frac{1}{(n-1)n} \quad , \quad n = 2, 3, \ldots$

$$\boxed{y = x + \sum_{n=2}^{\infty} \frac{x^n}{(n-1)n}}$$

3. naloga

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot 1 + \frac{\partial z}{\partial v} \cdot \frac{-y}{x^2}$$
$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot 0 + \frac{\partial z}{\partial v} \cdot \frac{1}{x}$$

$$x\left(\frac{\partial z}{\partial u} - \frac{y}{x^2}\frac{\partial z}{\partial v}\right) + y\left(\frac{1}{x}\frac{\partial z}{\partial v}\right) = z$$
$$u\frac{\partial z}{\partial u} = z$$

To *PDE* lahko rešujemo kot *navadno* diferencialno enačbo za neznanko $z(u, v) = f(u)$ in pri tem gledamo na v kot konstanto.

$$\frac{dz}{du} = \frac{z}{u}$$
$$\int \frac{dz}{z} = \int \frac{du}{u}$$
$$\ln z = \ln u + \ln C(v)$$
$$z = C(v)u$$

$$\boxed{z = C(\tfrac{y}{x})x}$$

4. naloga

$2y + 2e^x - (2y')' = 0$
$y'' - y = e^x$

$r^2 - 1 = 0$
$r_{1,2} = \pm 1$
$y_h = Ae^x + Be^{-x}$

$y_p = Ce^x x$
$y' = Ce^x(x+1)$
$y'' = Ce^x(x+2)$

$Ce^x(x+2) - Ce^x x = e^x$
$2Ce^x = e^x$
$C = \frac{1}{2}$

$y_p = \frac{x}{2}e^x$

$$\boxed{y = Ae^x + Be^{-x} + \tfrac{x}{2}e^x}$$

5. naloga

$$\int_{-\infty}^{\infty} p(x)\,dx = 1 \quad \rightarrow \quad A\int_0^1 x(1-x)\,dx = A\frac{1}{6} = 1 \quad \rightarrow \quad \boxed{A = 6}$$

$$P\big(X \leq \tfrac{1}{2} | \tfrac{1}{3} \leq X \leq \tfrac{2}{3}\big) = P\big(X \leq \tfrac{2}{3}\big) = \int_0^{\frac{2}{3}} 6x(1-x)\,dx = \boxed{\frac{20}{27}}$$