Izpit Matematika IV

6. februar 2014

1. S pomočjo *Laplaceove* transformacije rešite enačbo

$$y(t) = te^t - 2e^t \int_0^t e^{-u} y(u)\, du \quad .$$

2. S pomočo razvoja v *potenčno vrsto* rešite naslednjo diferencialno enačbo

$$y' = 3x^2 y \quad .$$

3. Z vpeljavo novih spremenljivk $v = x$, $z = x - y$ rešite parcialno diferencialno enačbo

$$u_{xx} + 2u_{xy} + u_{yy} = 0 \quad .$$

4. Poiščite *ekstremalo* funkcionala

$$I[y] = \int_a^b (y^2 - y'^2 - 2y \operatorname{ch} x)\, dx \quad .$$

5. Za diskretno slučajno spremenljivko, ki ima verjetnostno shemo

$$X : \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ \frac{1}{36} & \frac{3}{36} & \frac{5}{36} & \frac{7}{36} & \frac{9}{36} & \frac{11}{36} \end{pmatrix} \quad ,$$

izračunajte $D(X)$.

Rešitve

1. naloga

$$y(t) = te^t - 2\int_0^t e^{t-u} y(u)\, du$$

$$y(t) = te^t - 2e^t * y(t) \quad /\mathcal{L}$$
$$Y(s) = \frac{1}{(s-1)^2} - 2\,\frac{1}{s-1}\,Y(s)$$
$$Y(s)\left[1 + \frac{2}{s-1}\right] = \frac{1}{(s-1)^2}$$
$$Y(s) = \frac{1}{\left[1+\frac{2}{s-1}\right](s-1)^2} = \frac{1}{(s+1)(s-1)} = \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right)$$
$$y(t) = \boxed{\frac{1}{2}\left(e^t - e^{-t}\right)} = \boxed{\operatorname{sh} t}$$

2. naloga

$$y = \sum_{n=0}^{\infty} C_n x^n$$

$$\sum_{n=1}^{\infty} C_n n x^{n-1} = 3\sum_{n=0}^{\infty} C_n x^{n+2}$$

$$\sum_{n=0}^{\infty} C_{n+1}(n+1)x^n = \sum_{n=2}^{\infty} 3C_{n-2}x^n$$

$$n \geq 2 \quad \to \quad C_{n+1}(n+1) = 3C_{n-2} \quad \to \quad C_{n+1} = \tfrac{3}{n+1}C_{n-2}$$
$$n = 0 \quad \to \quad C_1 = 0 \quad \to \quad C_{3k+1} = 0$$
$$n = 1 \quad \to \quad C_2 = 0 \quad \to \quad C_{3k+2} = 0$$

C_0 poljuben

$$n+1 = 3k \quad \to \quad C_{3k} = \tfrac{3}{3k}C_{3k-3} = \tfrac{1}{k}C_{3k-3} \quad \to \quad C_{3k} = \tfrac{1}{k!}C_0$$

$$y = C_0 \sum_{n=0}^{\infty} \tfrac{1}{n!}x^{3n} = \boxed{C_0 e^{x^3}}$$

3. naloga

$$u_x = u_v \cdot 1 + u_z \cdot 1$$
$$u_y = u_v \cdot 0 + u_z \cdot (-1) = -u_z$$

$$u_{xx} = (u_v + u_z)_x = (u_v + u_z)_v \cdot 1 + (u_v + u_z)_z \cdot 1 = u_{vv} + 2u_{vz} + u_{zz}$$
$$u_{yy} = (-u_z)_y = (-u_z)_v \cdot 0 + (-u_z)_z \cdot (-1) = u_{zz}$$
$$u_{xy} = (u_x)_y = (u_v + u_z)_y = (u_v + u_z)_v \cdot 0 + (u_v + u_z)_z \cdot (-1) = -u_{vz} - u_{zz}$$

$$u_{xx} + 2u_{xy} + u_{yy} = 0$$
$$u_{vv} + 2u_{vz} + u_{zz} + 2(-u_{vz} - u_{zz}) + u_{zz} = 0$$
$$u_{vv} = 0$$

$$u_v = f(z)$$
$u = vf(z) + g(z) \quad , \quad f$ in g sta poljubni dvakrat odvedljivi funkciji

$$\boxed{u = xf(x-y) + g(x-y)}$$

4. naloga

$$f = y^2 - y'^2 - 2y \operatorname{ch} x$$
$$f_y - (f_{y'})' = 0$$
$$2y - 2\operatorname{ch} x - (-2y')' = 0$$
$$y'' + y = \operatorname{ch} x$$

$$r^2 + 1 = 0$$
$$r_{1,2} = \pm i$$
$$y_h = A\cos x + B\sin x$$

$$y_p = C\operatorname{ch} x + D\operatorname{sh} x$$
$$C\operatorname{ch} x + D\operatorname{sh} x + C\operatorname{ch} x + D\operatorname{sh} x = \operatorname{ch} x$$
$$C = \tfrac{1}{2}\,,\ D = 0$$

$$y = y_h + y_p = \boxed{A\cos x + B\sin x + \frac{1}{2}\operatorname{ch} x}$$

5. naloga

$$E(X) = 1\tfrac{1}{36} + 2\tfrac{3}{36} + 3\tfrac{5}{36} + 4\tfrac{7}{36} + 5\tfrac{9}{36} + 6\tfrac{11}{36} = \tfrac{1}{36}\Big(1 + 6 + 15 + 28 + 45 + 66\Big) = \tfrac{161}{36}$$

$$E(X^2) = 1\tfrac{1}{36} + 4\tfrac{3}{36} + 9\tfrac{5}{36} + 16\tfrac{7}{36} + 25\tfrac{9}{36} + 36\tfrac{11}{36} = \tfrac{1}{36}\Big(1 + 12 + 45 + 112 + 225 + 396\Big) = \tfrac{791}{36}$$

$$D(X) = E(X^2) - E^2(X) = \tfrac{791}{36} - \left(\tfrac{161}{36}\right)^2 = \boxed{\frac{2555}{1296}}$$