1. KOLOKVIJ MATEMATIKA IV

9.4.2010

skupina A

1. (40%) Z *Laplacovo* transformacijo poišči rešitev $x(t)$ diferencialne enačbe
$$\begin{aligned} x'' - 4x' + 13x &= 26 \\ x(0) &= 6 \\ x'(0) &= 5 \end{aligned}$$

2. (30%) Reši diferencialno enačbo
$$2y'' - xy'' - 2y' = 0$$
z nastavkom $y = \sum_{n=0}^{\infty} C_n x^n$!
   (a) (10%) Rekurzijska formula za koeficiente.
   (b) (10%) Izrazi rešitev z elementarnimi funkcijami pri pogojih
   $$\begin{aligned} y(0) &= 0 \\ y'(0) &= 1 \end{aligned}$$
   (c) (10%) Zapiši splošno rešitev enačbe !

3. (30%) Dana je diferencialna enačba
$$y'' - \frac{3}{x}y' + (4x^2 + \frac{3}{x^2})y = 0$$
   (a) (20%) Z vpeljavo neodvisne spremenljivke $z = x^2$ in funkcije $y = x^2 u$ reši enačbo !
   (b) (10%) Splošno rešitev enačbe izrazi z elementarnimi funkcijami !

skupina B

1. (40%) Z *Laplacovo* transformacijo poišči rešitev $x(t)$ diferencialne enačbe

$$\begin{aligned} x'' - 6x' + 13x &= 26 \\ x(0) &= 5 \\ x'(0) &= 5 \end{aligned}$$

2. (30%) Reši diferencialno enačbo

$$y'' - xy'' - 2y' = 0$$

z nastavkom $y = \sum_{n=0}^{\infty} C_n x^n$!

   (a) (10%) Rekurzijska formula za koeficiente.

   (b) (10%) Izrazi rešitev z elementarnimi funkcijami pri pogojih

$$\begin{aligned} y(0) &= 0 \\ y'(0) &= 1 \end{aligned}$$

   (c) (10%) Zapiši splošno rešitev enačbe !

3. (30%) Dana je diferencialna enačba

$$xy'' + 3y' + 4x^3 y = 0$$

   (a) (20%) Z vpeljavo neodvisne spremenljivke $z = x^2$ in funkcije $y = \frac{u}{x}$ reši enačbo !

   (b) (10%) Splošno rešitev enačbe izrazi z elementarnimi funkcijami !

Rešitve

skupina A

1. naloga

$$(s^2X - 6s - 5) - 4(sX - 6) + 13X = \frac{26}{s}$$

$$(s^2 - 4s + 13)X = 6s - 19 + \frac{26}{s}$$

$$X = \frac{6s^2 - 19s + 26}{s(s^2 - 4s + 13)} = \frac{A}{s} + \frac{Bs + C}{s^2 - 4s + 13}$$

$$6s^2 - 19s + 26 = A(s^2 - 4s + 13) + (Bs + C)s$$

$$6 = A + B$$

$$-19 = -4A + C$$

$$26 = 13A$$

$$A = 2\ ,\ B = 4\ ,\ C = -11$$

$$X = \frac{2}{s} + \frac{4s - 11}{(s-2)^2 + 9} = \frac{2}{s} + \frac{4(s-2) - 3}{(s-2)^2 + 9}$$

$$\boxed{x(t) = 2 + e^{2t}(4\cos 3t - \sin 3t)}$$

2. naloga

a)

$$y = \sum_{n=0}^{\infty} C_n x^n$$

$$y' = \sum_{n=1}^{\infty} C_n n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} C_n n(n-1) x^{n-2}$$

$$2\sum_{n=2}^{\infty} C_n n(n-1)x^{n-2} - \sum_{n=2}^{\infty} C_n n(n-1)x^{n-1} - 2\sum_{n=1}^{\infty} C_n n x^{n-1} = 0$$

$$2\sum_{n=1}^{\infty} C_{n+1}(n+1)n x^{n-1} - \sum_{n=1}^{\infty} C_n n(n-1)x^{n-1} - 2\sum_{n=1}^{\infty} C_n n x^{n-1} = 0$$

$$\sum_{n=1}^{\infty} \Big[2C_{n+1}(n+1)n - C_n n(n-1) - 2C_n n\Big] x^{n-1} = 0$$

$$2C_{n+1}(n+1)n - C_n\Big(n(n-1) + 2n\Big) = 0 \quad , \quad n = 1, 2, 3, \ldots$$

$$\boxed{C_{n+1} = \tfrac{1}{2}C_n \quad , \quad n = 1, 2, 3, \ldots}$$

b)

$$C_0 = 0 \quad , \quad C_1 = 1$$

$$y = x + \frac{1}{2}x^2 + \frac{1}{4}x^3 + \frac{1}{8}x^4 + \cdots + \frac{1}{2^{n+1}}x^n + \cdots$$

$$\boxed{y = \tfrac{x}{1-x/2}}$$

c)

splošna rešitev $\boxed{y = A + B\tfrac{x}{1-x/2}}$

3. naloga

a)

Označimo $u' = \frac{du}{dz}$

$$y' = 2xu + x^2u'2x = 2xu + 2x^3u'$$

$$y'' = 2u + 2xu'2x + 6x^2u' + 2x^3u''2x = 2u + 10x^2u' + 4x^4u''$$

$$2u + 10x^2u' + 4x^4u'' - \frac{3}{x}(2xu + 2x^3u') + (4x^2 + \frac{3}{x^2})x^2u = 0$$

$$4x^4u'' + 4x^2u' + (4x^4 - 1)u = 0$$

$$z^2u'' + zu' + (z^2 - \frac{1}{4})u = 0$$

$$u = \mathcal{J}_{\frac{1}{2}}(z)$$

$$\boxed{y = x^2\mathcal{J}_{\frac{1}{2}}(x^2)}$$

b)

splošna rešitev

$$y = x^2\Big[A\sqrt{\frac{2}{\pi x^2}}\sin(x^2) + B\sqrt{\frac{2}{\pi x^2}}\cos(x^2)\Big]$$

$$\boxed{\text{y}= Cx\sin x^2 + Dx\cos x^2}$$

Rešitve

skupina B

1. naloga

$$(s^2X - 5s - 5) - 6(sX - 5) + 13X = \frac{26}{s}$$

$$(s^2 - 6s + 13)X = 5s - 25 + \frac{26}{s}$$

$$X = \frac{5s^2 - 25s + 26}{s(s^2 - 6s + 13)} = \frac{A}{s} + \frac{Bs + C}{s^2 - 6s + 13}$$

$$5s^2 - 25s + 26 = A(s^2 - 6s + 13) + (Bs + C)s$$

$$5 = A + B$$

$$-25 = -6A + C$$

$$26 = 13A$$

$$A = 2\ ,\ B = 3\ ,\ C = -13$$

$$X = \frac{2}{s} + \frac{3s - 13}{(s-3)^2 + 4} = \frac{2}{s} + \frac{3(s-3) - 4}{(s-3)^2 + 4}$$

$$\boxed{x(t) = 2 + e^{3t}(3\cos 2t - 2\sin 2t)}$$

2. naloga

a)

$$y = \sum_{n=0}^{\infty} C_n x^n$$

$$y' = \sum_{n=1}^{\infty} C_n n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} C_n n(n-1) x^{n-2}$$

$$\sum_{n=2}^{\infty} C_n n(n-1)x^{n-2} - \sum_{n=2}^{\infty} C_n n(n-1)x^{n-1} - 2\sum_{n=1}^{\infty} C_n n x^{n-1} = 0$$

$$\sum_{n=1}^{\infty} C_{n+1}(n+1)n x^{n-1} - \sum_{n=1}^{\infty} C_n n(n-1)x^{n-1} - 2\sum_{n=1}^{\infty} C_n n x^{n-1} = 0$$

$$\sum_{n=1}^{\infty} \Big[C_{n+1}(n+1)n - C_n n(n-1) - 2C_n n\Big] x^{n-1} = 0$$

$$C_{n+1}(n+1)n - C_n\Big(n(n-1) + 2n\Big) = 0 \quad , \quad n = 1, 2, 3, \dots$$

$$\boxed{C_{n+1} = C_n \quad , \quad n = 1, 2, 3, \dots}$$

b)

$C_0 = 0 \quad , \quad C_1 = 1$

$y = x + x^2 + x^3 + x^4 + \cdots + x^n + \cdots$

$$\boxed{y = \tfrac{x}{1-x}}$$

c)

splošna rešitev $\boxed{y = A + B\tfrac{x}{1-x}}$

3. naloga

a)

Označimo $u' = \frac{du}{dz}$

$$y' = \frac{u'2xx - u}{x^2} = \frac{u'2x^2 - u}{x^2}$$

$$y'' = \frac{(u''2x2x^2 + u'4x - u'2x)x^2 - (u'2x^2 - u)2x}{x^4} = \frac{4x^5u'' - 2x^3u' + 2xu}{x^4}$$

$$\frac{4x^4u'' - 2x^2u' + 2u}{x^2} + \frac{6x^2u' - 3u}{x^2} + 4x^3\frac{u}{x} = 0$$

$$4x^4u'' - 2x^2u' + 2u + 6x^2u' - 3u + 4x^4u = 0$$

$$z^2u'' + zu' + (z^2 - \frac{1}{4})u = 0$$

$$u = \mathcal{J}_{\frac{1}{2}}(z)$$

$$\boxed{y = \tfrac{1}{x}\mathcal{J}_{\frac{1}{2}}(x^2)}$$

b)

splošna rešitev

$$y = \frac{1}{x}\Big[A\sqrt{\frac{2}{\pi x^2}}\sin(x^2) + B\sqrt{\frac{2}{\pi x^2}}\cos(x^2)\Big]$$

$$\boxed{\text{y}{=}C\,\frac{\sin x^2}{x^2} + D\,\frac{\cos x^2}{x^2}}$$