1. Kolokvij MATEMATIKA IV

12.4.2012

1. (30%) Z *Laplacovo* transformacijo poišči rešitev $y(t)$ diferencialne enačbe

$$\begin{aligned} y'' - 4y' + 8y &= 4 \\ y(0) &= 0 \\ y'(0) &= -4 \end{aligned}$$

2. (40%) Dana je diferencialna enačba

$$y'' + (1 - \frac{2}{x^2})y = 0$$

(a) (20%) Z vpeljavo funkcije $u(x)$, kjer je $y = u\sqrt{x}$, izrazi rešitev z *Besselovimi* funkcijami !

(b) (20%) Izrazi rešitev s trigonometričnimi funkcijami !

3. (30%) *Legendrovi* polinomi $P_n(x)$ so ortogonalni na intervalu $(-1, 1)$.

(a) (10%) Izračunaj

$$\int_{-1}^{1} [\,P_2(x) + 1\,][\,P_1(x) + 1\,]\,dx \quad !$$

(b) (10%) Določi konstanto k, tako da je

$$\int_{0}^{1} [\,P_2(x) + k\,]P_1(x)\,dx = 0 \quad !$$

(c) (10%) Izračunaj

$$\int_{0}^{1} [\,P_2(x) + 1\,][\,P_4(x) + 1\,]\,dx \quad !$$

$P_0 = 1\,,\ P_1 = x\,,\ P_2 = \frac{1}{2}(3x^2 - 1)\,,\ P_3 = \frac{1}{2}(5x^3 - 3x)\,,\ P_4 = \frac{1}{8}(35x^4 - 30x^2 + 3)\,,\ \ldots$

Rešitve

1. naloga

$$(s^2Y+4)-4(sY)+8Y=\frac{4}{s}$$

$$(s^2-4s+8)Y=\frac{4}{s}-4$$

$$Y=\frac{4-4s}{s(s^2-4s+8)}=\frac{A}{s}+\frac{Bs+C}{s^2-4s+8}$$

$$4-4s=A(s^2-4s+8)+(Bs+C)s$$

$$\begin{aligned} ks^2: &\quad A+B &= 0\\ ks: &\quad -4A+C &= -4\\ k: &\quad 8A &= 4 \end{aligned}$$

$$A=\frac{1}{2}\;,\;B=-\frac{1}{2}\;,\;C=-2$$

$$Y=\frac{1}{2}\;\frac{1}{s}+\frac{-\frac{1}{2}s-2}{s^2-4s+8}=\frac{1}{2}\;\frac{1}{s}+\frac{-\frac{1}{2}(s-2)-3}{s^2-4s+8}$$

$$Y=\frac{1}{2}\;\frac{1}{s}-\frac{1}{2}\;\frac{(s-2)}{(s-2)^2+4}-\frac{3}{2}\;\frac{2}{(s-2)^2+4}$$

$$\boxed{y(t)=\tfrac{1}{2}\Big[1-e^{2t}(\cos 2t+3\sin 2t)\Big]}$$

2. naloga

(a)

$$y = u\sqrt{x}$$

$$y' = u'\sqrt{x} + u\frac{1}{2\sqrt{x}}$$

$$y'' = u''\sqrt{x} + u'\frac{1}{2\sqrt{x}}2 - u\frac{1}{4\sqrt{x^3}}$$

$$u''\sqrt{x} + u'\frac{1}{\sqrt{x}} - u\frac{1}{4\sqrt{x^3}} + u\sqrt{x} - \frac{2u}{\sqrt{x^3}} = 0 \qquad / \cdot \sqrt{x^3}$$

$$x^2u'' + xu' + \left(x^2 - \frac{9}{4}\right)u = 0$$

$$u = C_1\mathcal{J}_{\frac{3}{2}}(x) + C_2\mathcal{J}_{-\frac{3}{2}}(x)$$

$$\boxed{y = \sqrt{x}\Big[C_1\mathcal{J}_{\frac{3}{2}}(x) + C_2\mathcal{J}_{-\frac{3}{2}}(x)\Big]}$$

(b)

$$\mathcal{J}_{n+1}(x) + \mathcal{J}_{n-1}(x) = \frac{2n}{x}\mathcal{J}_n(x)$$

$$\mathcal{J}_{n+1}(x) = \frac{2n}{x}\mathcal{J}_n(x) - \mathcal{J}_{n-1}(x)$$

$$n = \tfrac{1}{2} \quad \to \quad \mathcal{J}_{\frac{3}{2}}(x) = \tfrac{1}{x}\mathcal{J}_{\frac{1}{2}}(x) - \mathcal{J}_{-\frac{1}{2}}(x)$$

$$\mathcal{J}_{n-1}(x) = \frac{2n}{x}\mathcal{J}_n(x) - \mathcal{J}_{n+1}(x)$$

$$n = -\tfrac{1}{2} \quad \to \quad \mathcal{J}_{-\frac{3}{2}}(x) = \tfrac{-1}{x}\mathcal{J}_{-\frac{1}{2}}(x) - \mathcal{J}_{\frac{1}{2}}(x)$$

$$y = \sqrt{x}\Big[C_1\Big(\frac{1}{x}\mathcal{J}_{\frac{1}{2}}(x) - \mathcal{J}_{-\frac{1}{2}}(x)\Big) + C_2\Big(\frac{-1}{x}\mathcal{J}_{-\frac{1}{2}}(x) - \mathcal{J}_{\frac{1}{2}}(x)\Big)\Big]$$

$$y = \sqrt{x}\,\sqrt{\frac{2}{\pi x}}\Big[C_1\Big(\frac{\sin x}{x} - \cos x\Big) + C_2\Big(\frac{-\cos x}{x} - \sin x\Big)\Big]$$

Označimo $D_1 = \sqrt{\tfrac{2}{\pi}}C_1 \quad , \quad D_2 = -\sqrt{\tfrac{2}{\pi}}C_2$

$$\boxed{y = D_1\Big(\tfrac{\sin x}{x} - \cos x\Big) + D_2\Big(\tfrac{\cos x}{x} + \sin x\Big)}$$

3. naloga

(a)

$$\int_{-1}^{1} [P_2(x) + 1][P_1(x) + 1]\ dx \quad = \int_{-1}^{1} P_2(x)P_1(x)\ dx \quad + \int_{-1}^{1} P_0(x)P_1(x)\ dx \quad +$$

$$\int_{-1}^{1} P_2(x)P_0(x)\ dx\ + \int_{-1}^{1} dx\ = \boxed{2}$$

(b)

$$\int_{0}^{1} P_2(x)P_1(x)\ dx\ = \int_{0}^{1} \frac{1}{2}\Big(3x^2 - 1\Big)x\ dx\ = \int_{0}^{1}\Big(\frac{3}{2}x^3 - \frac{x}{2}\Big)\ dx\ = \frac{1}{8}$$

$$\int_{0}^{1} P_1(x)\ dx\ = \int_{0}^{1} x\ dx\ = \frac{1}{2}$$

$$\frac{1}{8} + k\frac{1}{2} = 0 \quad \rightarrow \quad \boxed{k = -\tfrac{1}{4}}$$

(c)

Pod integralom je soda funkcija, zato:

$$\int_{0}^{1} [P_2(x) + 1][P_4(x) + 1]\ dx\ = \frac{1}{2}\int_{-1}^{1} [P_2(x) + 1][P_4(x) + 1]\ dx\ =$$

$$\frac{1}{2}\int_{-1}^{1} P_2(x)P_4(x)dx + \frac{1}{2}\int_{-1}^{1} P_0(x)P_4(x)dx + \frac{1}{2}\int_{-1}^{1} P_2(x)P_0(x)dx + \frac{1}{2}\int_{-1}^{1} dx\ = \boxed{1}$$