2. Kolokvij MATEMATIKA IV

Bolonjski študij

7.junij 2013

1. (30%) Z enačbama $y = xu$, $z = x^2$ vpeljite v diferencialno enačbo

$$xy'' - y' + 4x^3y = 0$$

neodvisno spremenljivko z in funkcijo $u(z)$.

   (a) (20%) Zapišite vsaj eno rešitev diferencialne enačbe !

   (b) (10%) Zapišite tisto rešitev, za katero je $y(0) = 0$, $y''(0) = 1$!

2. (40%) Rešite *Dirichletovo* nalogo za pravokotnik:

$$\begin{aligned} u_{xx} + u_{yy} &= 0 \qquad , \quad 0 < x < \pi \ , \ 0 < y < 1 \\ u(0, y) &= 0 \\ u(\pi, y) &= 0 \\ u(x, 0) &= 0 \\ u(x, 1) &= \sin(2x) \quad ! \end{aligned}$$

3. (30%) Poiščite ekstremalo funkcionala

$$F(y) = \int_0^2 (xy' + y'^2)\, dx$$

$$\begin{aligned} y(0) &= 1 \\ y(2) &= 0 \quad ! \end{aligned}$$

2. Kolokvij MATEMATIKA IV

PredBolonjski študij

7.6.2013

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3. (30%) Vržemo tri kovance, slučajna spremenljivka $X =$ število padlih grbov.

   (a) (10%) Podajte verjetnostno funkcijo slučajne spremenljivke X !

   (b) (20%) Po prvem metu obdržimo kovance padle na grb, jih še enkrat vržemo in slučajna spremenljivka $Y =$ število grbov v tem metu. Podajte verjetnostno funkcijo slučajne spremenljivke Y !

Rešitve

1. naloga

V izpeljavi je $'$ (operator odvajanja) razumeti tako: $y' = \frac{dy}{dx}$, $u' = \frac{du}{dz}$

$$y' = u + xu'2x = u + 2x^2u'$$
$$y'' = u'2x + 4xu' + 2x^2u''2x = 6xu' + 4x^3u''$$

$$6x^2u' + 4x^4u'' - u - 2x^2u' + 4x^4u = 0 \quad / : 4 \quad , \quad x^2 \rightarrow z$$
$$z^2u'' + zu' + (z^2 - \tfrac{1}{4})u = 0$$

To je *Besselova* diferencialna enačba za $\nu = \frac{1}{2}$

a)

$$u = \mathcal{J}_{\frac{1}{2}}(z)$$
$$\boxed{y = x\mathcal{J}_{\frac{1}{2}}(x^2)}$$

b)

Splošna rešitev:

$$u = a\mathcal{J}_{\frac{1}{2}}(z) + b\mathcal{J}_{-\frac{1}{2}}(z) = \frac{1}{\sqrt{z}}(A\sin z + B\cos z)$$
$$y = A\sin(x^2) + B\cos(x^2)$$

$$y(0) = 0 \quad \rightarrow \quad B = 0$$

$$y' = A\cos(x^2)2x$$
$$y'' = A(-\sin(x^2)4x^2 + 2\cos(x^2))$$
$$y''(0) = 1 \quad \rightarrow \quad A = \frac{1}{2}$$

$$\boxed{y = \frac{1}{2}\sin(x^2)}$$

2. naloga

$$u(x,y) = F(x)G(y)$$

$$F''(x)G(y) = -F(x)G''(y)$$

$$\frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)} = -\lambda^2$$

$$\frac{F''(x)}{F(x)} = -\lambda^2$$

$$F''(x) + \lambda^2 F(x) = 0$$

$$k^2 + \lambda^2 = 0$$

$$k_{1,2} = \pm\lambda i$$

$$F(x) = A\cos(\lambda x) + B\sin(\lambda x)$$

$$x = 0 \quad \to \quad A = 0$$

$$x = \pi \quad \to \quad \sin(\lambda\pi) = 0 \quad \to \quad \lambda_n = n$$

$$F_n(x) = B_n \sin(nx)$$

$$\frac{G''(y)}{G(y)} = n^2$$

$$G''(y) - n^2 G(y) = 0$$

$$k^2 - n^2 = 0$$

$$k_{1,2} = \pm n$$

$$G_n(y) = C_n \operatorname{ch}(ny) + D_n \operatorname{sh}(ny)$$

$$u(x,y) = \sum_{n=1}^{\infty} \sin(nx)(c_n \operatorname{ch}(ny) + d_n \operatorname{sh}(ny))$$

$$y = 0 \quad \to \quad \sum_{n=1}^{\infty} c_n \sin(nx) = 0 \quad \to \quad c_n = 0$$

$$y = 1 \quad \to \quad \sum_{n=1}^{\infty} d_n \operatorname{sh}(n)\sin(nx) = \sin(2x) \quad \to \quad d_n = 0\ ,\ d_2 = \frac{1}{\operatorname{sh} 2}$$

$$\boxed{u(x,y) = \frac{1}{\operatorname{sh} 2}\sin(2x)\operatorname{sh}(2y)}$$

2. naloga - druga rešitev

$$u(x,y) = F(x)G(y)$$
$$\frac{F''(x)}{F(x)} = -\frac{G''(y)}{G(y)} = -\lambda^2$$

$$\frac{F''(x)}{F(x)} = -\lambda^2$$
$\cdots$
$$F_n(x) = B_n \sin(nx)$$

$$\frac{G''(y)}{G(y)} = n^2$$
$$G''(y) - n^2 G(y) = 0$$
$$k^2 - n^2 = 0$$
$$k_{1,2} = \pm n$$
$$G_n(y) = C_n e^{ny} + D_n e^{-ny}$$

$$u(x,y) = \sum_{n=1}^{\infty} \sin(nx)(c_n e^{ny} + d_n e^{-ny})$$
$$y = 0 \quad \rightarrow \quad \sum_{n=1}^{\infty} \sin(nx)(c_n + d_n) = 0$$
$$y = 1 \quad \rightarrow \quad \sum_{n=1}^{\infty} \sin(nx)(c_n e^{n} + d_n e^{-n}) = \sin(2x)$$

$$n \neq 2 \quad \rightarrow \quad c_n = d_n = 0$$
$$c_2 + d_2 = 0 \quad , \quad c_2 e^2 + d_2 e^{-2} = 1$$
Rešitev tega sistema enačb je
$$d_2 = -c_2 = \frac{1}{e^2 - e^{-2}}$$

$$\boxed{u(x,y) = \frac{1}{e^2 - e^{-2}} \sin(2x)(e^{2y} - e^{-2y})}$$

3. naloga

$$0 - (x + 2y')' = 0$$
$$x + 2y' = 2A$$
$$y' = A - \frac{x}{2}$$
$$y = Ax + B - \frac{x^2}{4}$$

$$x = 0 \quad \rightarrow \quad B = 1$$
$$x = 2 \quad \rightarrow \quad 2A + 1 - 1 = 0 \quad \rightarrow \quad A = 0$$

$$\boxed{y = 1 - \frac{x^2}{4}}$$

3. naloga

a)

$$X : \begin{pmatrix} 0 & 1 & 2 & 3 \\ \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{pmatrix}$$

b)

Označimo $X_i = (X = i)$, $Y_k = (Y = k)$ in uporabimo formulo

$$P(Y_k) = \sum_{i=0}^{3} P(X_i)P(Y_k/X_i)$$

$$P(Y_3) = P(X_3)P(Y_3/X_3) = \frac{1}{8} \cdot \frac{1}{8} = \frac{1}{64}$$

$$P(Y_2) = P(X_3)P(Y_2/X_3) + P(X_2)P(Y_2/X_2) = \frac{1}{8} \cdot \frac{3}{8} + \frac{3}{8} \cdot \frac{1}{4} = \frac{9}{64}$$

$$P(Y_1) = P(X_3)P(Y_1/X_3) + P(X_2)P(Y_1/X_2) + P(X_1)P(Y_1/X_1) = \frac{1}{8} \cdot \frac{3}{8} + \frac{3}{8} \cdot \frac{2}{4} + \frac{3}{8} \cdot \frac{1}{2} = \frac{27}{64}$$

$$P(Y_0) = P(X_3)P(Y_0/X_3) + P(X_2)P(Y_0/X_2) + P(X_1)P(Y_0/X_1) + P(X_0) = \frac{1}{8} \cdot \frac{1}{8} + \frac{3}{8} \cdot \frac{1}{4} + \frac{3}{8} \cdot \frac{1}{2} + \frac{1}{8} = \frac{27}{64}$$