2. Kolokvij MATEMATIKA IV

6.junij 2014

1. (30%) Poiščite rešitev $u(x,y)$ parcialne diferencialne enačbe

$$\begin{aligned} xu_{xy} + u_y &= y \\ u(x,1) &= \tfrac{1}{2} \\ u_y(1,y) &= 3y \quad . \end{aligned}$$

2. (40%) Z *Laplaceovo transformacijo* poiščite rešitev $u(x,t)$ parcialne diferencialne enačbe

$$\begin{aligned} x\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} &= xt \\ u(x,0) &= 0 \\ u(0,t) &= 0 \quad . \end{aligned}$$

3. (30%) Poiščite ekstremalo funkcionala

$$\begin{aligned} F(y) &= \int_0^{\ln 2} (y^2 + y'^2 + 2ye^x)\, dx \\ y(0) &= 3 \\ y(\ln 2) &= \ln 2 \quad . \end{aligned}$$

Rešitve

1. naloga

Vpeljemo neznako $u_y = v$

$xv_x + v = y \quad , \quad v(1,y) = 3y$

Rešimo kot *navadno* diferencialno enačbo

$xv' + v = y \quad , \quad v(1) = 3y$

Homogena:

$$x\frac{dv}{dx} = -v$$

$$\frac{dv}{v} = -\frac{dx}{x} \quad / \int$$

$$\ln v = -\ln x + \ln C$$

$$v_h = \frac{C}{x}$$

Variacija konstante:

$$v = \frac{C(x)}{x}$$

$$x(\frac{C'}{x} - \frac{C}{x^2}) + \frac{C}{x} = y$$

$$C' = y$$

$$C(x) = \int y dx = xy + K$$

$$v = y + \frac{K}{x}$$

$$v(0) = 3y \quad \to \quad 3y = y + K \quad \to \quad K = 2y$$

$$v = y + \frac{2y}{x}$$

$$u_y = y + \frac{2y}{x} \quad \to \quad u = \int (y + \frac{2y}{x}) dy = \frac{y^2}{2} + \frac{y^2}{x} + D$$

$$u(x,1) = \frac{1}{2} \quad \to \quad \frac{1}{2} = \frac{1}{2} + \frac{1}{x} + D \quad \to \quad D = -\frac{1}{x}$$

$$\boxed{u = \frac{y^2}{2} + \frac{y^2}{x} - \frac{1}{x}}$$

2. naloga

Označimo $\mathcal{L}[u(x,t)] = U(x,s) = U$

$$x\frac{\partial U}{\partial x} + sU = \frac{x}{s^2}$$

Rešimo kot *navadno* diferencialno enačbo

$$xU' + sU = \frac{x}{s^2} \quad , \quad U(0) = 0$$

Homogena:

$$x\frac{dU}{dx} = -sU$$

$$\frac{dU}{U} = -\frac{s}{x}dx \quad / \int$$

$$\ln U = -s\ln x + \ln C$$

$$U_h = \frac{C}{x^s}$$

Variacija konstante:

$$U = \frac{C(x)}{x^s}$$

$$x(\frac{C'}{x^s} - \frac{sC}{x^{s+1}}) + s\frac{C}{x^s} = \frac{x}{s^2}$$

$$C' = \frac{x^s}{s^2}$$

$$C = \frac{x^{s+1}}{s^2(s+1)} + D$$

$$U = \frac{x}{s^2(s+1)} + \frac{D}{x^s}$$

$$U(0) = 0 \quad \rightarrow \quad D = 0$$

$$\frac{1}{s^2(s+1)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+1}$$

$$1 = A(s+1) + Bs(s+1) + Cs^2$$

$$s = 0 \quad \rightarrow \quad A = 1$$

$$s = -1 \quad \rightarrow \quad C = 1$$

$$ks^2 \quad \rightarrow \quad B + C = 0 \quad \rightarrow \quad B = -1$$

$$U = x(\frac{1}{s^2} - \frac{1}{s} + \frac{1}{s+1})$$

$$\boxed{u(x,t) = x(t - 1 + e^{-t})}$$

3. naloga

Eulerjeva diferencialna enačba:

$$2y + 2e^x - (2y')' = 0$$

$$y'' - y = e^x$$

$$\lambda^2 - 1 = 0$$

$$\lambda_{1,2} \pm 1$$

$$y_h = Ae^x + Be^{-x}$$

$$y_p = Ce^x x$$

$$y' = Ce^x(x+1)$$

$$y'' = Ce^x(x+2)$$

$$C = \frac{1}{2}$$

$$y = y_h + y_p = Ae^x + Be^{-x} + \frac{1}{2}e^x x$$

$$x = 0 \quad \rightarrow \quad A + B = 3$$

$$x = \ln 2 \quad \rightarrow \quad 2A + \frac{1}{2}B + \ln 2 = \ln 2$$

$$A = -1\ ,\ B = 4$$

$$\boxed{y = 4e^{-x} - e^x + \frac{x}{2}e^x}$$