2. Kolokvij MATEMATIKA IV

12. junij 2015

1. (30%) Izrazite funkcijo $\mathcal{J}_1''(x)$ z *Besselovima* funkcijama $\mathcal{J}_0(x)$ in $\mathcal{J}_1(x)$.

2. (40%) Poiščite rešitev $u(x,t)$ parcialne diferencialne enačbe

$$\begin{aligned} u_t + u &= u_{xx} \\ u_x(0,t) &= 0 \\ u_x(\pi,t) &= 0 \\ u(x,0) &= 2\sin^2 x \quad . \end{aligned}$$

3. (30%) Poiščite ekstremalo funkcionala

$$\begin{aligned} F(y) &= \int_0^1 \sqrt{y(1+y'^2)}\, dx \\ y(0) &= \tfrac{1}{2} \\ y(1) &= \tfrac{1}{2} \quad . \end{aligned}$$

Rešitve

1. naloga

$$\mathcal{J}_1'' = (\mathcal{J}_1')' \ = \ \Big(-\frac{1}{x}\mathcal{J}_1 + \mathcal{J}_0\Big)' \ = \ \frac{1}{x^2}\mathcal{J}_1 - \frac{1}{x}\mathcal{J}_1' + \mathcal{J}_0' \ =$$

$$\frac{1}{x^2}\mathcal{J}_1 - \frac{1}{x}\Big(-\frac{1}{x}\mathcal{J}_1 + \mathcal{J}_0\Big) + \mathcal{J}_{-1} \ = \ \frac{1}{x^2}\mathcal{J}_1 + \frac{1}{x^2}\mathcal{J}_1 - \frac{1}{x}\mathcal{J}_0 - \mathcal{J}_1 \ =$$

$$\boxed{\Big(\frac{2}{x^2} - 1\Big)\mathcal{J}_1 - \frac{1}{x}\mathcal{J}_0}$$

2. naloga

$$u = F(x)G(t)$$
$$F(x)G'(t) + F(x)G(t) = F''(x)G(t)$$
$$\frac{G'(t)}{G(t)} + 1 = \frac{F''(x)}{F(x)} = -k^2$$

$$F''(x) + k^2 F(x) = 0$$
$$r^2 + k^2 = 0$$
$$r_{1,2} = \pm ki$$
$$F(x) = A\cos(kx) + B\sin(kx)$$
$$F'(x) = -Ak\sin(kx) + Bk\cos(kx)$$

$$x = 0 \quad \to \quad B = 0$$
$$x = \pi \quad \to \quad \sin(k\pi) = 0 \quad \to \quad k = n = 0, 1, 2, \ldots$$
$$F_n(x) = A_n \cos(nx)$$

$$\frac{G'(t)}{G(t)} + 1 = -n^2$$
$$\int \frac{dG}{G} = \int -(n^2+1)dt$$
$$\ln G = -(n^2+1)t + \ln C$$
$$G_n(t) = C_n e^{-(n^2+1)t}$$

$$u(x,t) = \sum_{n=0}^{\infty} b_n \cos(nx) e^{-(n^2+1)t}$$

$$t = 0 \quad \to \quad \sum_{n=0}^{\infty} b_n \cos(nx) = 2\sin^2 x = 1 - \cos 2x$$

$$b_0 = 1 \ , \ b_2 = -1 \ , \ b_n = 0$$

$$\boxed{u(x,t) = e^{-t} - e^{-5t}\cos 2x}$$

3. naloga

$$\sqrt{y(1+y'^2)} - y'\frac{\sqrt{y}\;2y'}{2\sqrt{1+y'^2}} \;=\; A$$

$$\sqrt{y}(1+y'^2-y'^2) \;=\; A\sqrt{1+y'^2}$$

$$y = A^2(1+y'^2) \quad , \quad A^2 = C$$

$$\frac{y}{C} - 1 \;=\; y'^2$$

$$\frac{dy}{dx} \;=\; \pm\sqrt{\frac{y}{C}-1} \qquad \text{opomba : druga rešitev je } y = C = \frac{1}{2}$$

$$\pm\int \frac{dy}{\sqrt{\frac{y}{C}-1}} \;=\; \int dx$$

$$\pm 2C\sqrt{\frac{y}{C}-1} \;=\; x + D$$

$$\frac{y}{C} - 1 \;=\; \left(\frac{x+D}{2C}\right)^2$$

$$y = \frac{(x+D)^2}{4C} + C$$

$$x = 0 \quad \rightarrow \quad \frac{D^2}{4C} + C \;=\; \frac{1}{2}$$

$$x = 1 \quad \rightarrow \quad \frac{(1+D)^2}{4C} + C \;=\; \frac{1}{2}$$

$$\frac{(1+D)^2 - D^2}{4C} \;=\; 0$$

$$1 + 2D = 0 \quad \rightarrow \quad D = -\frac{1}{2}$$

$$\frac{1}{16C} + C \;=\; \frac{1}{2}$$

$$(4C-1)^2 = 0 \quad \rightarrow \quad C = \frac{1}{4}$$

$$\boxed{y = x^2 - x + \frac{1}{2}}$$