Izpit Matematika I - VSP

28.januar 2014

1. Izračunajte
$$(\sqrt{3}+i)^3$$

2. Dan je splošni člen zaporedja
$$a_n = \frac{4n^2-4n}{(n+\sqrt{n})(2n+1)}$$
Izračunajte limito zaporedja in utemeljite odgovor.

3. Poiščite lokalni ekstrem funkcije
$$f(x) = x\ln x - x$$
in čimbolj natančno narišite graf $y = f(x)$.

4. Izračunajte določeni integral
$$\int_1^3 \frac{x+3}{x^2+3x+2}\,dx\,.$$

5. Lok parabole $y=\sqrt{x+2}$ v drugem kvadrantu zavrtimo okrog osi x. Izračunajte površino nastale vrtenine.

Rešitve

1. naloga

Enostavna rešitev :

$(\sqrt{3}+i)^3 = (\sqrt{3}+i)^2(\sqrt{3}+i) = (3+2\sqrt{3}\,i-1)(\sqrt{3}+i) =$
$(2+2\sqrt{3}\,i)(\sqrt{3}+i) = 2\sqrt{3}+6i+2i-2\sqrt{3} = \boxed{8i}$

Rešitev z *Moivreovo* formulo :

$z = (\sqrt{3}+i)$
$|z| = \sqrt{3+1} = 2$
$\operatorname{tg}\varphi = \dfrac{1}{\sqrt{3}} \quad , \quad \varphi = \dfrac{\pi}{6}$
$(\sqrt{3}+i)^3 = 2^3(\cos 3\dfrac{\pi}{6} + i\sin 3\dfrac{\pi}{6}) = 8(\cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}) = \boxed{8i}$

2. naloga

$$\lim_{n\to\infty} \frac{4n^2-4n}{(n+\sqrt{n})(2n+1)} =$$

$$\lim_{n\to\infty} \frac{4-\frac{4}{n}}{(1+\frac{\sqrt{n}}{n})(2+\frac{1}{n})} =$$

$$\frac{\lim\limits_{n\to\infty}(4-\frac{4}{n})}{\lim\limits_{n\to\infty}(1+\frac{1}{\sqrt{n}})\cdot\lim\limits_{n\to\infty}(2+\frac{1}{n})} = \frac{4}{1\cdot 2} = \boxed{2}$$

3. naloga

$f'(x) = \ln x + x\frac{1}{x} - 1 = \ln x$

Stacionarna točka je $x = 1$

$f''(x) = \frac{1}{x}$

$f''(1) = 1 > 0 \quad \rightarrow \quad x = 1$ je minimum

Definicijsko območje : $x > 0$

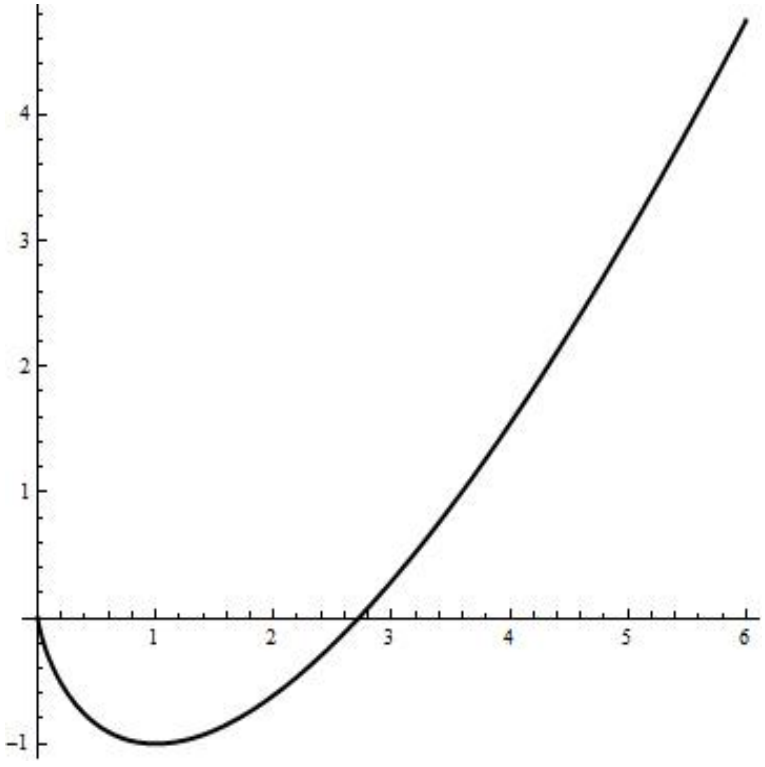
Ničle : $x(\ln x - 1) = 0 \quad \rightarrow \quad \ln x = 1 \quad \rightarrow \quad x = e$

Polov in asimptot ni

$\lim\limits_{x\to\infty} f(x) = \infty$

$\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} \frac{\ln x}{x^{-1}} - \lim\limits_{x\to 0} x = \lim\limits_{x\to 0} \frac{x^{-1}}{-x^{-2}} - 0 = \lim\limits_{x\to 0} -x = 0$

Točka ekstrema : $T(1. - 1)$



4. naloga

$$\frac{x+3}{x^2+3x+2} = \frac{A}{x+1} + \frac{B}{x+2} = \frac{A(x+2)+B(x+1)}{(x+1)(x+2)}$$

$$x+3 = (A+B)x + (2A+B)$$

$$A+B = 1\ ,\ 2A+B = 3$$

$$A = 2\ ,\ B = -1$$

$$\int_1^3 \frac{x+3}{x^2+3x+2}\,dx \ = \int_1^3 \Big(\frac{2}{x+1} - \frac{1}{x+2}\Big)\,dx \ = 2\ln|x+1| - \ln|x+2| \Big|_1^3 =$$

$$2\ln 4 - \ln 5 - 2\ln 2 + \ln 3 = \boxed{\ln\frac{12}{5}}$$

5. naloga

$$P = 2\pi \int_{-2}^0 \sqrt{x+2}\sqrt{1+\Big(\frac{1}{2\sqrt{x+2}}\Big)^2}\,dx \ = 2\pi \int_{-2}^0 \sqrt{x+2+\frac{1}{4}}\,dx \ =$$

$$\pi \int_{-2}^0 \sqrt{4x+9}\,dx \ = \pi \frac{(4x+9)^{3/2}}{4\frac{3}{2}} \Big|_{-2}^0 = \tfrac{\pi}{6}(27-1) = \boxed{\frac{13\pi}{3}}$$