DIFERENCIALNI IZPIT

8. september 2009

1. Podana je krivulja

$$\vec{r}(t) = (\sqrt{2}\operatorname{ch} t, \sin t + \cos t, \sin t - \cos t)$$

in točki $T_1(\sqrt{2}, 1, -1)$, $T_2\big(\frac{5\sqrt{2}}{4}, \sin(\log 2) + \cos(\log 2), \sin(\log 2) - \cos(\log 2)\big)$.

(a) Določite enačbo tangentne premice na krivuljo $\vec{r}(t)$ v točki T_1.

(b) Izračunajte dolžino loka krivulje $\vec{r}(t)$ med točkama T_1 in T_2.

Rešitev.

(a) Točka T_1 je jasno dosežena pri $t = 0$. Tako dobimo

$$\begin{aligned}\dot{\vec{r}}(t) &= (\sqrt{2}\operatorname{sh} t, \cos t - \sin t, \cos t + \sin t)\\ \dot{\vec{r}}(0) &= (0, 1, 1)\end{aligned}$$

Enačba iskane premice se tako glasi $x = \sqrt{2}$, $y - 1 = z + 1$.

(b) Točkama T_1 in T_2 pripadata $t_1 = 0$ in $t_2 = \log 2$. Tako dobimo

$$\begin{aligned} ds &= \sqrt{\dot{\vec{r}}(t)\cdot\dot{\vec{r}}(t)}\,dt = \sqrt{(\sqrt{2}\operatorname{sh} t)^2 + (\cos t - \sin t)^2 + (\cos t + \sin t))^2}\,dt = \\ &= ... = \sqrt{2\operatorname{sh}^2 t + 2}\,dt = \sqrt{2(\operatorname{sh}^2 t + 1)}\,dt = \sqrt{2\operatorname{ch}^2 t}\,dt = \sqrt{2}\operatorname{ch} t\,dt \\ s &= \int_0^{\log 2} ds = \int_0^{\log 2} \sqrt{2}\operatorname{ch} t\,dt = \sqrt{2}\operatorname{sh} t\Big|_0^{\log 2} = \sqrt{2}\,\frac{e^{\log 2} - e^{-\log 2}}{2} = \\ &= \sqrt{2}\,\frac{2 - \frac{1}{2}}{2} = \frac{3\sqrt{2}}{4}\end{aligned}$$

2. Določite parameter a tako, da bo krivuljni integral

$$\int_C \left(2x - \frac{a\cos z}{x}, \frac{z}{1+y^2z^2} + \frac{2\cos z}{y}, \frac{y}{1+y^2z^2} - (a^2-2)\log(xy)\sin z\right)\cdot d\vec{r}$$

neodvisen od poti in ga za primer, ko je C poljubna krivulja od točke $T_1(1, 1, 1)$ do točke $T_2(2, \frac{1}{2}, 0)$, izračunajte.

Rešitev. Vemo, da je krivuljni integral $\int_C \vec{V}\cdot d\vec{r}$ neodvisen od poti, ko velja $\operatorname{rot}\vec{V} = \vec{0}$. Torej mora v našem primeru veljati

$$\left(-\frac{(a^2-4)\sin z}{y}, \frac{(a^2+a-2)\sin z}{x}, 0\right) = (0, 0, 0),$$

kar je res, ko velja $a^2 - 4 = (a-2)(a+2) = 0$ in $a^2 + a - 2 = (a+2)(a-2) = 0$. Tako dobimo $a = -2$.

Za izračun dobljenega integrala najprej izračunamo potencial vektorskega polja $\vec{V}$:

$$\begin{aligned}
\int \left(2x + \frac{2\cos z}{x}\right) dx &= x^2 + 2\cos z \log x + D(y,z) \\
\int \left(\frac{z}{1+y^2z^2} + \frac{2\cos z}{y}\right) dy &= \arctan(yz) + 2\cos z \log y + D(x,z) \\
\int \left(\frac{y}{1+y^2z^2} - 2\log(xy)\sin z\right) dz &= \arctan(yz) + 2\log xy \cos z + D(x,y) \\
u &= x^2 + 2\log xy\cos z + \arctan(yz) + D
\end{aligned}$$

(Upoštevali smo dejstvo $\log x + \log y = \log xy$.) Vemo, da je iskani integral enak

$$u(T_2) - u(T_1) = 4 - \left(1 + \frac{\pi}{4}\right) = 3 - \frac{\pi}{4}$$

3. Izračunajte kompleksni integral

$$\int_{|z-2-2i|=\frac{5}{2}} \frac{8}{z(z-2)^3(z-2i)}\, dz,$$

kjer je integracija v pozitivni smeri.

Rešitev. Iz teorije vemo, da je dovolj gledati residuume v singularnostih znotraj integracijskega območja, torej le v $z = 2$ in $z = 2i$. Označimo $f(z) := \frac{8}{z(z-2)^3(z-2i)}$.

$$\begin{aligned}
\operatorname{res}_{z=2i} f(z) &= \lim_{z\to 2i} f(z)(z-2i) = \lim_{z\to 2i} \frac{8}{z(z-2)^3} = \\
&= \frac{8}{2i(2i-2)^3} = \frac{1}{2i(i-1)^3} = \frac{1}{-4+4i} = \frac{-4-4i}{16+16} = -\frac{1}{8} - \frac{i}{8} \\
\operatorname{res}_{z=2} f(z) &= \frac{1}{2}\lim_{z\to 2} \left(f(z)(z-2)^3\right)'' = \frac{1}{2}\lim_{z\to 2}\left(\frac{8}{z(z-2i)}\right)'' = \frac{1}{2}\lim_{z\to 2}\left(\frac{-16z+16i}{(z^2-2iz)^2}\right)' \\
&= \frac{1}{2}\lim_{z\to 2} \frac{-16(z^2-2iz)^2 - (-16z+16i)2(z^2-2iz)(2z-2i)}{(z^2-2iz)^4} = \\
&= \frac{1}{2}\,\frac{-16(4-4i)^2 - (-32+16i)2(4-4i)(4-2i)}{(4-4i)^4} = \\
&= \frac{1}{2}\,\frac{-(1-i)^2 - (-2+i)(1-i)(2-i)}{(1-i)^4} = \frac{-(1-i) - (-2+i)(2-i)}{2(1-i)^3} = \\
&= \ldots = \frac{2-3i}{-4-4i} = \frac{(2-3i)(-4+4i)}{16+16} = \ldots = \frac{1}{8} + \frac{5i}{8}
\end{aligned}$$

Iskani integral je tako enak

$$\int_{|z-2-2i|=\frac{5}{2}} f(z)\,dz = 2\pi i\big(\operatorname{res}_{z=2i} f(z) + \operatorname{res}_{z=2} f(z)\big) =$$
$$= 2\pi i\left(-\frac{1}{8} - \frac{i}{8} + \frac{1}{8} + \frac{5i}{8}\right) = 2\pi i\,\frac{i}{2} =$$
$$= -\pi$$

4. Z *Laplace*-ovo transformacijo poiščite rešitev $x(t)$ diferencialne enačbe

$$\begin{aligned} x''' - 4x'' + 4x' &= 0 \\ x(0) &= 0 \\ x'(0) &= 1 \\ x''(0) &= 2 \end{aligned}$$

Rešitev.

$$\begin{aligned} s^3X - s - 2 - 4(s^2X - 1) + 4sX &= 0 \\ (s^3 - 4s^2 + 4s)X &= s - 2 \\ X &= \frac{s-2}{s(s^2-4s+4)} = \\ &= \frac{1}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2} \\ 1 &= A(s-2) + Bs \quad \Longrightarrow \quad a = -\frac{1}{2},\ b = \frac{1}{2} \\ x(t) &= \frac{1}{2}\big(e^{2t} - 1\big) \end{aligned}$$

5. Poiščite rešitev $u(x,t)$ parcialne diferencialne enačbe

$$\begin{aligned} u_{tt} &= 9u_{xx}, \quad 0 < x < 3,\ 0 < t \\ u(0,t) &= 0 \\ u(3,t) &= 0 \\ u(x,0) &= 0 \\ u_t(x,0) &= \sin\left(\frac{\pi x}{3}\right) + \sin(\pi x) \end{aligned}$$

Rešitev.

$u(x,t) = F(x)G(t)$

$F(x)G''(t) = 9F''(x)G(t)$

$$\frac{G''(t)}{G(t)} = 9\frac{F''(x)}{F(x)} = -\lambda^2$$
$$u(0,t) = 0 \Longrightarrow F(0) = 0$$
$$u(3,t) = 0 \Longrightarrow F(3) = 0$$

$$9F''(x) + \lambda^2 F(x) = 0$$
$$9r^2 + \lambda^2 = 0$$
$$r_{1,2} = \pm\frac{\lambda}{3}i$$
$$F(x) = A\cos\left(\frac{\lambda}{3}x\right) + B\sin\left(\frac{\lambda}{3}x\right)$$
$$F(0) = 0 \Longrightarrow A = 0$$
$$F(3) = 0 \Longrightarrow \lambda = n\pi$$
$$F_n(x) = B_n\sin\left(\frac{n\pi}{3}x\right)$$

$$\frac{G''(t)}{G(t)} = -(n\pi)^2$$
$$G''(t) + (n\pi)^2 G(t) = 0$$
$$r^2 + (n\pi)^2 = 0$$
$$r_{1,2} = \pm n\pi i$$
$$G_n(t) = C_n\cos(n\pi t) + D_n\sin(n\pi t)$$

$$u(x,t) = \sum_{n=1}^{\infty}\Big(c_n\cos(n\pi t) + d_n\sin(n\pi t)\Big)\sin\left(\frac{n\pi}{3}x\right)$$
$$u(x,0) = 0 \Longrightarrow \sum_{n=1}^{\infty} c_n\sin\left(\frac{n\pi}{3}x\right) \Longrightarrow c_n = 0$$
$$u_t(x,t) = \sum_{n=1}^{\infty}\Big(-c_n n\pi\sin(n\pi t) + d_n n\pi\cos(n\pi t)\Big)\sin\left(\frac{n\pi}{3}x\right)$$
$$u_t(x,0) = \sin\left(\frac{\pi x}{3}\right) + \sin(\pi x) = \sum_{n=1}^{\infty} d_n n\pi\sin\left(\frac{n\pi}{3}x\right)$$
$$d_1 = \frac{1}{\pi}\ ,\ d_3 = \frac{1}{3\pi}\ ,\ \text{ ostali } d_n = 0$$
$$u(x,t) = \frac{1}{\pi}\Big(\sin(\pi t)\sin\left(\frac{\pi x}{3}\right) + \frac{1}{3}\sin(3\pi t)\sin(\pi x)\Big)$$